Summer course: Plasmonic resonances

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Outline of Part I

- Nanoparticles?
- Spectral problem
- Quasistatic approximation
- Biosensing problem
- Neumann Poincaré operator
- Poincaré variational problem

Nanoparticles?

[A. Moores and F. Goettmann, New J. Chem., 2006, 30, 1121-1132.]

- the first syntheses of metallic small particles date back to the 4th or 5th century BC where gold specimen were reported in China and Egypt.
- their optical properties were used for coloration of glass, ceramics, china and pottery

Nanoparticles?

- the interesting diffractive properties of these particles are linked to resonances phenomena
- plasmon resonances may occur in metallic particles if
  - the dielectric permittivity inside the particle is negative
  - the wavelength of the incident excitation is much larger than the dimension of the particle
- for nanoscale metallic particles, these resonances occur in the optical frequency range and they result in an extremely large enhancement of the electromagnetic field near the boundary of the particles
- this phenomena has applications in many areas such as nanophotonics, nanolithography, near field microscopy, biosensors and medicine (for example gold nanoparticles allow heat from infrared lasers to be targeted on cancer tumors www.understandingnano.com/nanoparticles.html)

Nanoparticles?

The desired resonance frequencies as well as the local fields enhancement can be achieved by controlling the geometry of the metallic nanostructure.

- they are filled with a real metal: the electric permittivity ε depends on the frequency of the excitation ω.
- their size δ is very small compare to the incident wavelength $\lambda = 2\pi/\omega$, that is $\lambda = \delta\omega/2\pi \prec\prec 1$. In practice δ is between 10 and 100 nm and $\lambda \sim 650$ nm.
- the corresponding plasmonic resonances are in the the optical frequency range.

The spectral problem

Let Ω be a bounded $C^{1,\alpha}$-domain in $\mathbb{R}^2$, with normalized size, that is $|\Omega| = 1$. Assume that the metallic nanoparticle occupies $\Omega_\delta = z + \delta\Omega$, where the position z is fixed in $\mathbb{R}^2$.
The complex number ω is said to be *a resonant frequency of the nanoparticle* Ω_δ if there exists a non-trivial solution E to the system (TE polarization):

$$\left\{ \begin{array}{lcl} \Delta E + \omega^2 \varepsilon(\omega, x)\mu_0 E & = & 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{\Omega_\delta} \cup \Omega_\delta, \\ \lfloor \varepsilon E \rfloor & = & 0 \quad \text{on} \quad \partial\Omega_\delta, \\ \left\lfloor \frac{\partial E}{\partial \nu} \right\rfloor & = & 0 \quad \text{on} \quad \partial\Omega_\delta, \end{array} \right.$$

where the electric permittivity ε is given by

$$\varepsilon(\omega, x) = \left\{ \begin{array}{ll} \varepsilon_0 & \text{for} \quad x \in \mathbb{R}^2 \setminus \overline{\Omega_\delta}, \\ \varepsilon_0 \hat{\epsilon}(\omega) & \text{for} \quad x \in \Omega_\delta, \end{array} \right.$$

The spectral problem

Drude model

The metal that fills the nanoparticle is assumed to be ¡¡real¿¿: Its dielectric constant is described by the Drude model:

$$\hat{\epsilon}(\omega) \quad = \quad \varepsilon_\infty - \frac{\omega_P^2}{\omega^2 + i\omega\Gamma},$$

where $\varepsilon_\infty > 0$, $\omega_P > 0$ and $\Gamma > 0$ are the metal parameters that are usually fitted using experiment data

[A. Moores and F. Goettmann, *The plasmon band in noble nanoparticles: an introduction to theory and applications*, New J. Chem., 2006, 30, 1121-1132.]

For example the function $\hat{\varepsilon}(\omega)$ with effective parameters: $\varepsilon_\infty = 3.7\ eV$, $\omega_P = 8.9\ eV$, $\Gamma = 0.021\ eV$ for silver reproduce quite well the experimental values of the dielectric constant in the frequency range $0.8\ eV$ to $4\ eV$

[P.B. Johnson and R. W. Christy, *Optical constants of the noble metals*, Phys. Rev. B, 6, 4370-4379 (1972).]

Quasi-static approximation

Since δ is too small an asymptotic analysis needs to be performed to avoid costly and complex computations.

Making the change of variables $\tilde{x} = z + \delta x$ in the system, we obtain

$$\begin{cases} \Delta \widetilde{E} + \delta^2 \omega^2 \tilde{\varepsilon}(\omega, x) \mu_0 \widetilde{E} & = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{\Omega} \cup \Omega, \\ \left[\tilde{\varepsilon} \widetilde{E} \right] & = 0 \quad \text{on} \quad \partial\Omega, \\ \left[\frac{\partial \widetilde{E}}{\partial \nu} \right] & = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where the electric permittivity $\tilde{\varepsilon}$ is given by

$$\tilde{\varepsilon}(\omega, x) = \begin{cases} \varepsilon_0 & \text{for} \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \\ \varepsilon_0 \hat{\epsilon}(\omega) & \text{for} \quad x \in \Omega, \end{cases}$$

Quasi-static approximation

When $\delta \to 0$, we obtain the spectral problem

$$\begin{cases} \Delta \widetilde{E} & = & 0 & \text{in} & \mathbb{R}^2 \setminus \overline{\Omega} \cup \Omega, \\ \left[\tilde{\varepsilon} \widetilde{E} \right] & = & 0 & \text{on} & \partial\Omega, \\ \left[\frac{\partial \widetilde{E}}{\partial \nu} \right] & = & 0 & \text{on} & \partial\Omega. \end{cases}$$

This was observed by [ID Mayergoyz, DR Fredkin, Z Zhang - Physical Review B, 2005].

$$\omega_j(\delta) \to \omega_0 \qquad \delta \to 0,$$

where ω_0 is a complex value for which there exists a non-trivial solution $\tilde{E}$ to the above problem.

Assume that $V = \tilde{\varepsilon}(\omega_0, x)E$, then

$$\nabla . \frac{1}{\tilde{\varepsilon}} \nabla V \quad = \quad 0 \qquad \text{in} \quad \mathbb{R}^2.$$

Consequently the condition $\Re \tilde{(\varepsilon)} = \Re(\varepsilon_0 \hat{\epsilon}(\omega_0)) < 0$ is necessary to have a solution.

Quasi-static approximation

The Quasi-static approximation is rigorously justified and the complete asymptotic expansion of the plasmonic resonances is derived.

Theorem (BT 15)

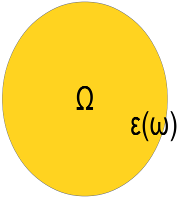
Let ω_0 *be a the unique quasistatic resonance in* $B_\rho(\omega_0) \subset \mathbb{C} \setminus \mathbb{R}_-$*, and denote by* m *its multiplicity. Then, there exists* $\delta_0 > 0$ *such that for* $0 < \delta \leq \delta_0$*, the set of values of the* ω_0*-group, satisfy*

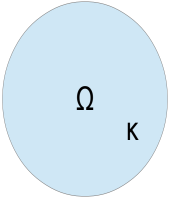
$$\begin{aligned} \frac{1}{m}\sum_{j=1}^{m} \omega_j(\delta) \quad &= \quad \omega_0 + \sum_{p=1}^{\infty} \omega_{p,0}^{(2)} \frac{1}{(\log(\delta))^p} + \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \omega_{p,n}^{(1)} \frac{\delta^{2n+1}}{(\log(\delta))^{p-n-1}} \\ &\quad + \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} \omega_{p,n}^{(2)} \frac{\delta^{2n}}{(\log(\delta))^{p-n}}, \end{aligned}$$

holds for $\delta \in (0, \delta_0)$*.*

Quasi-static approximation

Why metallic nanoparticles?



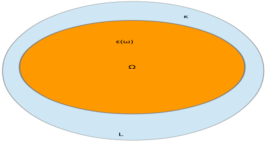


- size $\delta << \lambda$.
- permittivity $\varepsilon(\omega)$.
- plasmonic resonances $\frac{2\pi}{\omega_j(\delta)} \to \frac{2\pi}{\omega_0}$ as $\delta \to 0$.
- the wavelength $\frac{2\pi}{\omega_0}$ is within the optical range.

- size $\delta << \lambda$.
- permittivity is a constant κ.
- scattering resonances $\frac{2\pi}{\omega_j(\delta)} \to 0$ as $\delta \to 0$.
- the resonances can not be measured.

Biosensing problem

- nanoparticle:
  $\Omega_\delta = z + \delta\Omega$
- thin layer: $L_\delta = z + \delta L$
  Denote
  $\Omega_\delta^e = \mathbb{R}^2 \setminus (\Omega_\delta \cup \overline{L}_\delta)$
-



$$
\varepsilon_p(\omega, x) = \begin{cases} \varepsilon_0 & \text{for} \quad x \in \Omega_\delta^e, \\ \varepsilon_0 \kappa & \text{for} \quad x \in L_\delta, \\ \varepsilon_0 \hat{\epsilon}(\omega) & \text{for} \quad x \in \Omega_\delta, \end{cases}
$$

We consider the spectral problem:

$$
\begin{cases} \Delta E + \omega^2 \varepsilon_p(\omega, x) \mu_0 E = 0 & \text{in} \quad \Omega_\delta^e \cup L_\delta \cup \Omega_\delta, \\ \left\lfloor \varepsilon_p E \right\rfloor = 0 & \text{on} \quad \partial L_\delta \cup \partial \Omega_\delta, \\ \left\lfloor \frac{\partial E}{\partial \nu} \right\rfloor = 0 & \text{on} \quad \partial L_\delta \cup \partial \Omega_\delta, \end{cases}
$$

Let $\omega_{p,j}(\delta)$ be the plasmonic resonances of the perturbed nanoparticle.

Biosensing problem

Biosensing problem: to determine $\kappa(x)$ from the knowledge of the schift in plasmonic resonances: $(\omega_{p,j}(\delta) - \omega_j(\delta))$.

Making again the change of variables $\tilde{x} = z + \delta x$ in the system, we obtain

$$\begin{cases} \Delta \widetilde{E} + \delta^2 \omega^2 \tilde{\varepsilon}_p(\omega, x) \mu_0 \widetilde{E} & = 0 \quad \text{in} \quad \Omega^e \cup L \cup \Omega, \\ \left[\tilde{\varepsilon}_p \widetilde{E} \right] & = 0 \quad \text{on} \quad \partial L \cup \partial \Omega, \\ \left[\frac{\partial \widetilde{E}}{\partial \nu} \right] & = 0 \quad \text{on} \quad \partial L \cup \partial \Omega, \end{cases}$$

where the electric permittivity $\tilde{\varepsilon}_p$ is given by

$$\tilde{\varepsilon}_p(\omega, x) = \begin{cases} \varepsilon_0 & \text{for} \quad x \in \Omega^e, \\ \varepsilon_0 \kappa & \text{for} \quad x \in L, \\ \varepsilon_0 \hat{\epsilon}(\omega) & \text{for} \quad x \in \Omega, \end{cases}$$

Biosensing problem

Quasi-static approximation

The quasi-static approximation gives

$$\begin{cases} \Delta \widetilde{E} & = 0 \quad \text{in} \quad \Omega^e \cup L \cup \Omega, \\ \left[\tilde{\varepsilon}_p \widetilde{E} \right] & = 0 \quad \text{on} \quad \partial L \cup \partial \Omega, \\ \left[\frac{\partial \widetilde{E}}{\partial \nu} \right] & = 0 \quad \text{on} \quad \partial L \cup \partial \Omega, \end{cases}$$

Let $\omega_{L,j}(\kappa)$ be the quasi-static plasmonic resonances of the perturbed nanoparticule.

We neglect in the rest of the talk the error in the quasi-static approximation.

Biosensing problem: to determine κ from the knowledge of the schift in plasmonic resonances: $(\omega_{L,j}(\kappa) - \omega_{0,j})$.

Neumann-Poincaré operator

Recall that

$$\Delta_x \left(\frac{1}{2\pi} \log(|x-y|) \right) = \delta_y(x).$$

We now define, the single and double layer potentials of the functions $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\psi \in H^{\frac{1}{2}}(\partial\Omega)$ respectively, by

$$\begin{aligned} \mathcal{S}_0[\varphi](x) &= \frac{1}{2\pi} \int_{\partial\Omega} \log(|x-y|)\varphi(y)d\sigma(y), \\ \mathcal{D}_0[\psi](x) &= \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y)\cdot\nu(y)}{|x-y|^2}\psi(y)d\sigma(y), \end{aligned}$$

for $x \in \mathbb{R}^2 \setminus \partial\Omega$. We also define the boundary integral operator $\mathcal{K}_0$ on $H^{\frac{1}{2}}(\partial\Omega)$ by

$$\mathcal{K}_0[\psi](x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y)\cdot\nu(y)}{|x-y|^2}\psi(y)d\sigma(y),$$

for $x \in \partial\Omega$.

Neumann-Poincaré operator

Let $\mathcal{K}_0^*$ be the adjoint of $\mathcal{K}_0$, that is

$$\mathcal{K}_0^*[\varphi](x) \;=\; \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot\nu(x)}{|x-y|^2}\varphi(y)d\sigma(y),$$

for $x \in \partial\Omega,\ \varphi \in H^{-\frac{1}{2}}(\partial\Omega)$.

$\mathcal{K}_0^*$ is called the **Neumann-Poincaré** operator (appears in many applications with different names).

$$\begin{aligned}
\mathcal{S}_0[\varphi](x)|_+ &= \mathcal{S}_0[\varphi](x)|_-, \\
\left(\mathcal{D}_0[\psi](x)\right)|_\pm &= \left(\mp\frac{1}{2}I + \mathcal{K}_0\right)[\psi](x), \\
\left(\frac{\partial}{\partial\nu}\mathcal{S}_0[\varphi](x)\right)|_\pm &= \left(\pm\frac{1}{2}I + \mathcal{K}_0^*\right)[\varphi](x),
\end{aligned}$$

for $x \in \partial\Omega$.

Neumann-Poincaré operator

Lemma

$\mathcal{K}_0^*: H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is compact.

Proof.

Since Ω is a $C^{1,\alpha}$ domain, we have

$$\left|\frac{(x-y)\cdot\nu(x)}{|x-y|^2}\right| \leq C(\Omega)\frac{1}{|x-y|^{1-\alpha}} \text{ for } x, y \in \partial\Omega,\ x \neq y.$$

□

Lemma

The operator $\mathcal{A}: H^{-\frac{1}{2}}(\partial\Omega) \times \mathbb{R} \to H^{\frac{1}{2}}(\partial\Omega) \times \mathbb{R}$, defined by

$$\mathcal{A}(\varphi, a) = \left(\mathcal{S}_0[\varphi] + a, \int_{\partial\Omega} \varphi d\sigma\right),$$

is invertible.

Neumann-Poincaré operator

Lemma

Let (φ_e, a_e) *in* $H^{-\frac{1}{2}}(\partial\Omega)\times\mathbb{R}$, *be the unique solution of*

$$\mathcal{A}(\varphi_e, a_e) = \left(\mathcal{S}_0[\varphi_e] + a_e, \int_{\partial\Omega}\varphi_e d\sigma\right) = (0,1).$$

Then

i $\mathcal{S}_0 : \; H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ *is invertible if and only if* $a_e \neq 0$.

Define

$$H_0^{-\frac{1}{2}}(\partial\Omega) = \{\varphi \in H^{-\frac{1}{2}}(\partial\Omega),\ \langle\varphi, 1\rangle_{-\frac{1}{2},\frac{1}{2}} = 0\},$$
$$H_e^{\frac{1}{2}}(\partial\Omega) = \{\psi \in H^{\frac{1}{2}}(\partial\Omega),\ \langle\varphi_e, \psi\rangle_{-\frac{1}{2},\frac{1}{2}} = 0\}.$$

ii $\mathcal{S}_0 : \; H_0^{-\frac{1}{2}}(\partial\Omega) \to H_e^{\frac{1}{2}}(\partial\Omega)$ *is negative invertible operator.*

Neumann-Poincaré operator

Lemma

Let λ be a real number. The operator $\lambda I + \mathcal{K}_0^*$ is one to one on $H_0^{-\frac{1}{2}}(\partial\Omega)$ if $|\lambda| \geq \frac{1}{2}$, and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$, $\lambda I + \mathcal{K}_0^*$ is one to one on $H^{-\frac{1}{2}}(\partial\Omega)$.

Proof.

The argument is by contradiction. Assume that $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$, and $(\lambda I + \mathcal{K}_0^*)\,[\varphi] = 0$ for $\varphi \neq 0$ in $H_0^{-\frac{1}{2}}(\partial\Omega)$.

By Green formula we have $\mathcal{K}_0[1] = \frac{1}{2}$. We deduce that $(\lambda - \frac{1}{2})\langle \varphi, 1\rangle_{-\frac{1}{2},\frac{1}{2}} = 0$. Hence $\langle \varphi, 1\rangle_{-\frac{1}{2},\frac{1}{2}} = 0$, and

$$0 < A := \int_{\Omega} |\nabla \mathcal{S}_0[\varphi]|^2 \, dx, \;\; B := \int_{\mathbb{R}^2 \setminus \overline{\Omega}} |\nabla \mathcal{S}_0[\varphi]|^2 \, dx < \infty.$$

□

Neumann-Poincaré operator

Proof.

On the other hand, using the jump relations, we have

$$A = \int_{\partial\Omega} \left(-\frac{1}{2} I + \mathcal{K}_0^* \right) [\varphi] \mathcal{S}_0[\varphi] d\sigma,$$
$$B = -\int_{\partial\Omega} \left(\frac{1}{2} I + \mathcal{K}_0^* \right) [\varphi] \mathcal{S}_0[\varphi] d\sigma.$$

Consequently

$$\lambda = \frac{B - A}{2(B + A)}.$$

Thus, $|\lambda| < \frac{1}{2}$, which is in contradiction with the assumption.

If $\lambda = \frac{1}{2}$, then $A = 0$ and hence $\mathcal{S}_0[\varphi] = c$ on Ω. Since $\varphi \in H_0^{-\frac{1}{2}}(\partial\Omega)$, the constant $c \in H_e^{\frac{1}{2}}(\partial\Omega)$. Hence $c = 0$, and finally $\varphi = 0$.

□

Neumann-Poincaré operator

$\mathcal{K}_0^* : H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$, is compact, has a real discrete spectrum in $(-1/2, 1/2]$, that is symmetric with respect to 0, with only 0 as an accumulation point (only in dimension two).

We denote

$$\lambda_\infty = 0 < \cdots \leq \lambda_1^+ \leq \lambda_0 = \frac{1}{2},$$

the eigenvalues in $[0, \frac{1}{2}]$, repeated according to their multiplicity, and

$$\lambda_1^- = -\lambda_1^+ < \lambda_2^- = -\lambda_2^+ < \cdots < 0,$$

the eigenvalues in $(-\frac{1}{2}, 0)$, repeated according to their multiplicity.

$$\sigma_p(\mathcal{K}_0^*) \quad = \quad 0, 1/2,\ \lambda_j^\pm,\ j \geq 1.$$

Finally, we have the following symmetrization

$$\mathcal{S}_0\mathcal{K}_0^* \quad = \quad \mathcal{K}_0\mathcal{S}_0.$$

Neumann-Poincaré operator

One can introduce a new scalar product on $H_0^{-\frac{1}{2}}(\partial\Omega)$:

$$\langle \varphi, \psi \rangle_S := \langle \varphi, -\mathcal{S}_0[\psi] \rangle_{-\frac{1}{2},\frac{1}{2}},$$

for which $\mathcal{K}_0^*$ becomes selfadjoint [Khavinson-Putinar-Shapiro 09].
We denote φ_0, $\varphi_j^{\pm}$ the normalized eigenfunctions of $\mathcal{K}_0$ associated respectively to $1/2$ and $\lambda_j^{\pm}$, $j \geq 1$, that is

$$\begin{cases} \mathcal{K}_0^*[\varphi_j^{\pm}] & = & \lambda_j^{\pm}\varphi_j^{\pm} \\ \|\varphi_j^{\pm}\|_S & = & 1. \end{cases}$$

We also have the following spectral decomposition on $H_0^{-\frac{1}{2}}(\partial\Omega)$:

$$(\lambda I - \mathcal{K}_0^*)^{-1} = \frac{1}{\lambda}\mathcal{Q}_0 + \sum_{j=1}^{\infty} \frac{\langle \cdot, \varphi_j^{\pm} \rangle_S \varphi_j^{\pm}}{\lambda - \lambda_j^{\pm}}.$$

We now define the variational eigenvalues

$$k_0 = -\infty, \quad k_\infty = -1,$$

$$k_j^{\pm} := \frac{2\lambda_j^{\pm} + 1}{2\lambda_j^{\pm} - 1} \qquad \text{for } j \geq 1.$$

Neumann-Poincaré operator

Back now to the quasi-static approximation.

$$V(x) \; = \; \mathcal{S}_0[\varphi](x),$$

is a generalized eigenfunction if and only if $\varphi \in H^{-1/2}(\partial\Omega)$ satisfies

$$\left(\lambda(\omega) I - \mathcal{K}_0^*\right)[\varphi](x) \; = \; 0 \qquad x \in \partial\Omega,$$

where

$$\lambda(\omega) \; = \; \frac{\hat{\varepsilon}(\omega) + 1}{2(\hat{\varepsilon}(\omega) - 1)}.$$

The quasi-static resonances $(\omega_{0,j})_j$ are solutions to the dispersion relations $\hat{\epsilon}(\omega) \; = \; k_j^{\pm}, \; 0$ for $j \in \{\infty\} \cup \mathbb{N}^*$, and are explicitly given by

$$\pm\sqrt{\frac{\omega_p^2}{\varepsilon_\infty - k_j^{\pm}} - \frac{\Gamma^2}{4}} - i\frac{\Gamma}{2} \qquad \text{if} \quad k_j^{\pm} \geq \varepsilon_\infty - 4\frac{\omega_P^2}{\Gamma^2},$$

$$i\left(-\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \frac{\omega_p^2}{\varepsilon_\infty - k_j^{\pm}}}\right) \qquad \text{if} \quad k_j^{\pm} < \varepsilon_\infty - 4\frac{\omega_P^2}{\Gamma^2},$$

and the value $-i\Gamma$ corresponding to $\hat{\epsilon}(\omega) \; = \; 0$.

Poincaré variational problem

The space

$$W_0^{1,-1}(\mathbb{R}^2) = \left\{ \frac{u(x)}{(1+|x|^2)^{1/2}\ln(2+|x|^2)} \in L^2,\ \nabla u \in L^2, u(x) \to 0,\ \ |x| \to \infty \right\},$$

equipped with the scalar product

$$(u,v)_W := \int_{\mathbb{R}^2} \nabla u \cdot \nabla v dx,$$

is a Hilbert space.

Define $T : W_0^{1,-1}(\mathbb{R}^2) \to W_0^{1,-1}(\mathbb{R}^2)$ by

$$\forall\ v \in W_0^{1,-1}(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} \nabla Tu \cdot \nabla v dx \quad = \quad \int_{\Omega} \nabla u \cdot \nabla v dx.$$

Poincaré variational problem

The operator $T : W_0^{1,-1}(\mathbb{R}^2) \to W_0^{1,-1}(\mathbb{R}^2)$ is self-adjoint and bounded with with norm $\|T\| \leq 1$.

The spectral Poincaré variational problem is to find $(w, \beta) \in W_0^{1,-1}(\mathbb{R}^2) \setminus \{0\} \times \mathbb{R}$, solution to

$$\beta \int_{\mathbb{R}^2} \nabla w \cdot \nabla v dx \;=\; \int_{\Omega} \nabla w \cdot \nabla v dx, \qquad \forall\, v \in W_0^{1,-1}(\mathbb{R}^2).$$

Integrating by parts, one immediately obtains that any eigenfunction w is harmonic in Ω and in Ω', and satisfies the transmission conditions on $\partial\Omega$:

$$w|_+ = w|_-, \qquad \frac{\partial w}{\partial \nu}|_+ = (1 - \frac{1}{\beta})\frac{\partial w}{\partial \nu}|_-$$

Poincaré variational problem

Taking $k = 1 - \frac{1}{\beta}$, and

$$\gamma(x) = \begin{cases} 1 & \text{for} \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \\ k & \text{for} \quad x \in \Omega, \end{cases}$$

we obtain that (w, k) is a solution to the system

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla w) = 0 & \text{in } \mathbb{R}^2 \\ w \to 0 & \text{as } |x| \to \infty, \end{cases}$$

Here $\gamma(x)$ can be seen as the function $\frac{1}{\tilde{\varepsilon}(\omega_0, x)}$ in the quasistatic approximation, defined by

$$\tilde{\varepsilon}(\omega, x) = \begin{cases} \varepsilon_0 & \text{for} \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \\ \varepsilon_0 \hat{\epsilon}(\omega) & \text{for} \quad x \in \Omega. \end{cases}$$

Poincaré variational problem

$$\begin{aligned}\mathfrak{H}_S &= \{S_0[\varphi], \quad \varphi \in H^{-1/2}(\partial\Omega), \quad \int_{\partial\Omega} \varphi d\sigma = 0\} \\ &= \{u \in W_0^{1,-1}(\mathbb{R}^2), \quad \Delta u = 0 \text{ in } \Omega \cup \Omega', \quad [u]_{|\partial D} = 0\}\end{aligned}$$

$\mathfrak{H}_S$ is the subspace of single layer potentials in $W_0^{1,-1}(\mathbb{R}^2)$.

Lemma

The following assertions hold.

- i *The eigenspace of T associated to the eigenvalue $\beta = 1$ is $Ker(I - T) = \{v|_{\Omega'} = 0,\ v|_{\Omega} \in H_0^1(\Omega)\}$. This eigenspace does not contains any element of $\mathfrak{H}_S$ except $v = 0$.*
- ii *The eigenspace of T associated to the eigenvalue $\beta = 0$ is $Ker(T) = \{v|_{\Omega'} \in W_0^{1,-1}(\Omega'),\ v|_{\Omega} = 0\} \cup \mathbb{R}$.*

Poincaré variational problem

We can easily verify $T\mathfrak{H}_S \subset \mathfrak{H}_S$. We further keep the notation T for the restriction of T to $\mathfrak{H}_S$.

Lemma

We have $T = \frac{1}{2}I + R$, with $R : \mathfrak{H}_S \to \mathfrak{H}_S$ is compact.

Proof.

$$2\int_{R^2} \nabla Ru(X)\nabla v(X)dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}^+ + \frac{\partial u}{\partial \nu}^- \right) v(x)dx.$$

Since $u \in \mathfrak{H}_S$, we can write $u = \mathcal{S}_0[\varphi]$, with $\varphi \in H_0^{-\frac{1}{2}}(\partial\Omega)$. On the other hand, we have

$$\frac{\partial u}{\partial \nu}|^+ + \frac{\partial u}{\partial u}^- = 2K_0^*[\phi] = 2K_0^*\left[i_{H^{-1/2}} o \mathbb{S}_0^{-1}\left[(u|_{\partial\Omega}, 0)\right]\right](X),$$

where $i_{H^{-1/2}} : H^{-\frac{1}{2}}(\partial\Omega) \times \mathbb{R} \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is defined by $i_{H^{-1/2}}(f, a) = f$.

Poincaré variational problem

Then T is a Fredholm operator of index zero and its spectrum is real, discrete, contained in $(0,1)$ and symmetric with respect to $\frac{1}{2}$, with only $\frac{1}{2}$ as an accumulation point.

We denote $(\beta_n^{\pm})_{n\geq 1}$ the eigenvalues of T, ordered as follows:

$$0 < \beta_1^+ \leq \beta_2^+ \leq \cdots \leq \frac{1}{2},$$

the eigenvalues in $(0,1/2]$ and, similarly,

$$1 > \beta_1^- \geq \beta_2^- \geq \cdots \geq \frac{1}{2},$$

the eigenvalues in $[1/2,1)$. The eigenvalue $1/2$ is the unique accumulation point of the spectrum.

Poincaré variational problem

Lemma

Let $(w_n^{\pm})_{n\geq 1}$ be the eigenfunctions associated to $(\beta_n^{\pm})_{n\geq 1}$. Then

$$\begin{aligned}
\beta_n^+ &= \max_{u\in\mathfrak{H}_S,\,\perp w_1^+,\cdots,w_n^+} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx} \\
&= \min_{\substack{F_n\subset\mathfrak{H}_S\\ dim(F_n)=n}} \max_{u\in F_n} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx}
\end{aligned}$$

and similarly

$$\begin{aligned}
\beta_n^- &= \min_{u\in\mathfrak{H}_S,\,\perp w_1^-,\cdots,w_n^-} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx}, \\
&= \max_{\substack{F_n\subset\mathfrak{H}_S\\ dim(F_n)=n}} \min_{u\in F_n} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx}.
\end{aligned}$$

Poincaré variational problem

Lemma

Let u be in $\mathfrak{H}_S$. Then

$$u(x) = \sum_{n\geq 1} u_n^{\pm} w_n^{\pm}(x), \qquad x \in \mathbb{R}^2,$$

where

$$u_n^{\pm} = \frac{\int_{\mathbb{R}^2} \nabla u(x) \nabla w_n^{\pm}(x) dx}{\int_{\mathbb{R}^2} \left| \nabla w_n^{\pm}(x) \right|^2 dx}.$$

The series is convergent with respect to the norm $\| \cdot \|_W$.

Outline of Part II

- Motivation
- The Neumann-Poincaré operator
- The case of disks
- Inclusions with $C^{1+\alpha}$ boundaries
- Numerical illustration
- Conclusion

Motivation

Let $D_1, D_2 \subset \mathbb{R}^2$ be two bounded smooth inclusions separated by a distance $\delta > 0$, and

$$\gamma(X) = \begin{cases} k & X \in D_1 \cup D_2 \\ 1 & X \in \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}, \end{cases}$$

where $k \neq 1$ is a given real constant.

Define the potential u solution to the PDE

$$\begin{cases} \mathrm{div}(\gamma(X)\nabla u(X)) & = & 0 & \text{in } \mathbb{R}^2 \\ u(X) - H(X) & \to & 0 & \text{as } |X| \to \infty, \end{cases}$$

where H is a given harmonic function ($\Delta H(X) = 0$ in $\mathbb{R}^2$).

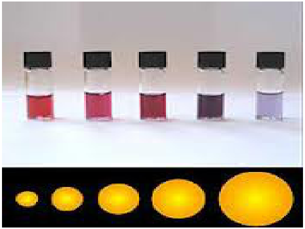
Motivation

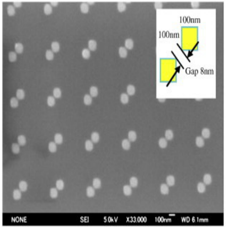
High contrast composite material ($k \in [0,1) \cup (1,+\infty]$)

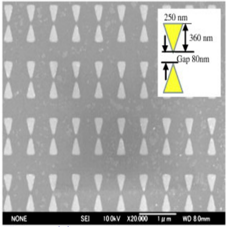
- Are the gradients ∇u uniformly bounded as $\delta \to 0$?
- Do the bounds depend on the contrast k?
- Does the presence of narrow regions in between in favor stress concentration that could lead to fracture?

Plasmonic resonances ($k < 0$ and $H = 0$)

- Plasmonic resonances of metallic nano-particles are the solutions to $\varepsilon(\omega) = k < 0$; How do the resonances behave when $\delta \to 0$?
- How the electric fields are confined inbetween the nanoparticles?



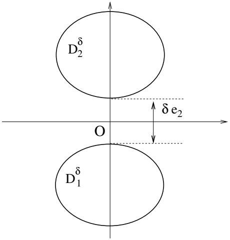




A model problem

Assume that D_1 and D_2 are the translates of 2 reference touching inclusions D_1^0 and D_2^0, that is

$$D_1 = D_1^0 + (0, \delta/2), \quad D_1 = D_1^0 + (0, -\delta/2).$$



- D_1^0 and D_2^0 are strictly convex and only meet at 0
- Γ_1 and Γ_2 have regularity $\mathcal{C}^{1,\alpha}$, for some $0 < \alpha < 1$
- Around 0, Γ_i is parametrized by a curve $x \longrightarrow (x, (-1)^i[\psi_i(x) + \delta/2])$
- $\psi_1(x) + \psi_2(x) \sim C|x|^m \quad$ as $|x| \to 0$

Related results

Case of a non degenerate contrast ($k \neq 0, \infty$)

• 2 disks in 2D for a conduction problem [Bonnetier-Vogelius 01]

$$||u||_{W^{1,\infty}(\Omega)} \;\leq\; C$$

• Piecewise Hölder conductivities scalar equations and strongly elliptic systems
[Li-Vogelius 01, Li-Nirenberg 03]

$$\Omega \;=\; D_0 \cup \left(\cup_{i=1}^{N} D_i \right)$$

Each D_i has $C^{1,\alpha}$ boundary
$0 < \Lambda \leq \gamma \leq \Lambda^{-1}$ and $\gamma_{|D_i}$ has regularity $C^{0,\mu}$

$$\sum_{i=0}^{N} ||u||_{C^{1,\alpha'}(D_i \cap \Omega_\varepsilon)} \;\leq\; C\; (||u|_{\partial\Omega}||)$$

where $\alpha' = \inf(\mu, \frac{\alpha}{2(1+\alpha)})$ and $C = C(\Omega, \varepsilon, \Lambda, N, \alpha, \mu)$ is independent of δ

Related results

Case of a possibly degenerate contrast ($k \geq 0$)

• 2 disks in 2D

[Ammari-Kang-Lim 05, Ammari-Kang-Lee-Lee-Lim 07, Ammari-Kang-Lee-Lim-Zribi 2010]

2 disks at a distance δ, of same radius r and conductivity k, meeting at $X = 0$ tangentially to the direction T.
Then, $u(X) = u_r(X) + u_s(X)$ with $||\nabla u_r|| \leq C$ where $C > 0$ is independent of δ, and k.

$$\text{if } k < 1 \quad \begin{cases} ||\nabla u_s||_\infty \leq \dfrac{C_1|\nabla H(0)\cdot T|}{2k + \sqrt{\delta/r}} \\ |\nabla u_s(0)| \geq \frac{C_2|\nabla H(0)\cdot T|}{2k+\sqrt{\delta/r}} \end{cases}$$

$$\text{if } k > 1 \quad \begin{cases} ||\nabla u_s||_\infty \leq \frac{C_1|\nabla H(0)\cdot N|}{2k^{-1}+\sqrt{\delta/r}} \\ |\nabla u_s(0)| \geq \frac{C_2|\nabla H(0)\cdot N|}{2k^{-1}+\sqrt{\delta/r}} \end{cases}$$

The Neumann-Poincaré operator

Let

$$G(X, Y) = \frac{1}{2\pi} \ln |X - Y|$$

be the Green function of Laplace operator, and

$$S_i f(X) = \int_{\Gamma_i} G(X, Y) f(Y)\, ds$$

The potential u can be expressed as

$$u(X) = S_1\varphi_1(X) + S_2\varphi_2(X) + H(X)$$

Let $\lambda = \frac{k+1}{2(k-1)} \in \mathbb{R} \setminus [-1/2, 1/2]$ and let $K_i^{*,\delta}$ denote the operator defined on $H^{-1/2}(\Gamma_i)$ by

$$K_i^{*,\delta} f(X) = \frac{1}{2\pi} \int_{\Gamma_i} \frac{(X - Y) \cdot \nu_i(X)}{|X - Y|^2} f(Y)\, ds_Y.$$

The Neumann-Poincaré operator

The layer potentials solve

$$(\lambda I - K^{*\delta}) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \partial_\nu H_{|\Gamma_1} \\ \partial_\nu H_{|\Gamma_2} \end{pmatrix}$$

where $K^{*\delta} = \begin{pmatrix} K_1^{*,\delta} & L_2^\delta \\ L_1^\delta & K_2^{*,\delta} \end{pmatrix}$ is the Neumann-Poincaré operator and where for $(\varphi_1, \varphi_2) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$,

$$\begin{cases} L_2^\delta \varphi_2(X) = -\dfrac{\partial}{\partial \nu_1} S_2 \varphi_2(X) & X \in \Gamma_1 \\ L_1^\delta \varphi_1(X) = -\dfrac{\partial}{\partial \nu_2} S_1 \varphi_1(X) & X \in \Gamma_2 \end{cases}$$

Classical potential theory: When $\delta > 0$, $\lambda I - K^{*,\delta}$ is a continuous linear mapping on $\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$, invertible with bounded inverse, for any $0 < \alpha < 1$ and for $|\lambda| > 1/2$

In the integral equation, the two parameters δ and k are separated

The Neumann-Poincaré operator

When $\delta > 0$, the operators $K^{*,\delta}$ are compact, but not self-adjoint
However, due to the Calderón identity

$$SK^{\delta} \;=\; K^{*,\delta}S$$

where

$$S\left(\begin{array}{c}\varphi_1\\ \varphi_2\end{array}\right) = \left(\begin{array}{c}(S_1\varphi_1 + S_2\varphi_2)_{|\Gamma_1}\\ (S_1\varphi_1 + S_2\varphi_2)_{|\Gamma_2}\end{array}\right)$$

one can introduce a new scalar product in an appropriate subspace of L^2

$$< \varphi, \psi >_S \;=\; < -S\varphi, \psi >_{L^2} = -\int_{\Gamma_1} S_1[\varphi_1]\psi_1 - \int_{\Gamma_2} S_2[\varphi_2]\psi_2$$

for which $K^{*\delta}$ is also self-adjoint.
[Carleman, Krein, Khavinson-Putinar-Shapiro]

The Neumann-Poincaré operator

Consequently, if $\lambda_n^\delta \in (-1/2, 1/2], \varphi_n^\delta$ denote the eigenelements of $K^{*,\delta}$ and if the data decomposes as

$$\begin{pmatrix} \partial_\nu H_{|\Gamma_1} \\ \partial_\nu H_{|\Gamma_2} \end{pmatrix} = \sum_{n\geq 1} \alpha_n \varphi_n^\delta$$

then one can write the solution of the integral equation $(\lambda I - K^{*\delta})\varphi = \partial_\nu H$ as

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \sum_{n\geq 1} \frac{\alpha_n}{\lambda - \lambda_n^\delta} \begin{pmatrix} \varphi_{1,n}^\delta \\ \varphi_{2,n}^\delta \end{pmatrix}$$

with convergence in the sense of the scalar product $< \cdot, \cdot >_S$, and

$$\nabla u = \nabla S_1 \varphi_1 + \nabla S_2 \varphi_2 + \nabla H$$

The singularities of φ are likely to depend on how $\lambda - \lambda_n^\delta$ becomes small as $\delta \to 0$ and $\lambda \to \pm 1/2$.

Uniform bounds for a non-degenerate contrast

Theorem (ABTV)

Let $0 < k \neq 0, \infty$ *be fixed and* Ω *be a bounded open domain containing* $D_1 \cup D_2$*. The solution* u *is bounded in* $\mathcal{C}^{1,\alpha_0}(\overline{\Omega \setminus (D_1 \cup D_2)}) \cap \mathcal{C}^{1,\alpha_0}(\overline{D_1}) \cap \mathcal{C}^{1,\alpha_0}(\overline{D_2})$*, uniformly with respect to* δ*, for any* $0 < \alpha_0 < \alpha$*.*

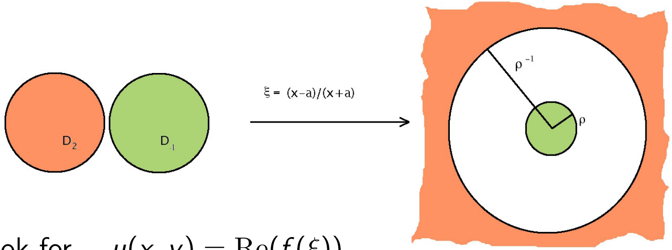
H. Ammari, E. Bonnetier, F, Triki and M. Vogelius. *Elliptic estimates in composite media with smooth inclusions: an integral equation approach,* Ann. Sci. Éc. Norm. Supér. volume 48, no. 2, p50, (2015).

The case of 2 disks centered at $(-1-\delta, 0)$ and $(1+\delta, 0)$

[see also Ammari-Ciraolo-Kang-Lee-Milton, McPhedran]

One can Transform into 2 concentric disks of radii ρ and $1/\rho$ with the conformal mapping

$$\xi = \frac{z-a}{z+a} \qquad a = \sqrt{\delta(2+\delta)} \qquad \rho = \frac{a-\delta}{a+\delta}$$



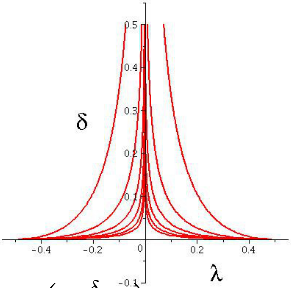
and look for $u(x,y) = \mathrm{Re}(f(\xi))$

One can explicitly compute the eigenelements of $K^{*,\delta}$:

$$k_n^- = -\left(\frac{1+\rho^{2n}}{1-\rho^{2n}}\right) \qquad \lambda - \lambda_n^- = \lambda - (1/2 - n\sqrt{2}\sqrt{\delta} + O(\delta))$$

$$k_n^+ = -\left(\frac{1-\rho^{2n}}{1+\rho^{2n}}\right) \qquad \lambda - \lambda_n^+ = \lambda - (-1/2 + n\sqrt{2}\sqrt{\delta} + O(\delta))$$

where $\rho = \dfrac{\sqrt{\delta(2+\delta)} - \delta}{\sqrt{\delta(2+\delta)} + \delta}$



and the expansion $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \sum_{n\geq 1} \frac{\alpha_n}{\lambda - \lambda_n^\delta} \begin{pmatrix} \varphi_{1,n}^\delta \\ \varphi_{2,n}^\delta \end{pmatrix}$ converges pointwise

(so that one recovers the pointwise estimates on ∇u_δ shown before)

Inclusions with $\mathcal{C}^{1+\alpha}$ boundaries

A min-max principle

Let $D = D_1 \cup D_2, \quad D' = \mathbb{R}^2 \setminus \overline{D}$

$$W_0^{1,-1}(\mathbb{R}^2) = \left\{ \frac{u(X)}{(1+|X|^2)^{1/2}\ln(2+|X|^2)} \in L^2,\ \nabla u \in L^2, u(X) \to 0,\ \ |X| \to \infty \right\}$$

$$\begin{aligned} \mathcal{H}_S &= \{S_1\varphi_1 + S_2\varphi_2, \quad \varphi_i \in H^{-1/2}(\Gamma_i), \quad \int_{\Gamma_1} \varphi_1 + \int_{\Gamma_2} \varphi_2 = 0\} \\ &= \{u \in W_0^{1,-1}(\mathbb{R}^2), \quad \Delta u = 0 \text{ in } D \cup D', \quad [u]_{|\partial D} = 0\} \end{aligned}$$

$\mathcal{H}_S$ is the subspace of single layer potentials in $W_0^{1,-1}(\mathbb{R}^2)$.

The operator T_δ defined by

$$\forall\ v \in W_0^{1,-1}(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} \nabla T_\delta u \cdot \nabla v \quad = \quad \int_D \nabla u \cdot \nabla v$$

is self-adjoint and satisfies $||T_\delta|| \leq 1$. It is also Fredholm of index 0

If $(w,\beta) \in \mathcal{H}_S \times \mathbb{R}$ is an eigenpair of T_δ, then

$$\forall\, v \in W_0^{1,-1}(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} \nabla T_\delta w \cdot \nabla v \;=\; \beta \int_{\mathbb{R}^2} \nabla w \cdot \nabla v \;=\; \int_D \nabla w \cdot \nabla v$$

so that $\nabla w = 0$ in $D \cup D'$ and

$$[w]_{|\Gamma_1 \cup \Gamma_2} \;=\; 0 \qquad \partial_n w^+ \;=\; (1 - 1/\beta)\partial_n w^-$$

w is thus solution to

$$\begin{cases} \operatorname{div}(a_\delta(X)\nabla w(X)) \;=\; 0 \quad \text{in } \mathbb{R}^2 \\ w \to 0 \quad \text{as } |X| \to \infty \end{cases}$$

with $a_\delta = 1_D + (1 - 1/\beta)1_{D'}$. In other words,
$w = S_1\varphi_1 + S_2\varphi_2$ with

$$(\lambda I - K^{*,\delta}) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \;=\; 0$$

and $\lambda \;=\; \dfrac{k+1}{2(k-1)} \;=\; 1/2 - \beta$ is an eigenvalue of $K^{*,\delta}$

As a consequence, the eigenvalues of $K^{*,\delta}$ are given by the min-max principle

$$\beta_n^{\delta,-} = \max_{\substack{F_n \subset \mathcal{H}_S \\ \dim(F_n) = n}} \min_{u \in F_n \setminus \{0\}} \frac{\int_D |\nabla u|^2}{\int_{\mathbb{R}^2} |\nabla u|^2}$$

$$\beta_n^{\delta,+} = \min_{\substack{F_n \subset \mathcal{H}_S \\ \dim(F_n) = n}} \max_{u \in F_n \setminus \{0\}} \frac{\int_D |\nabla u|^2}{\int_{\mathbb{R}^2} |\nabla u|^2}$$

so that, the nondegenerate eigenvalues of $K^{*,\delta}$ satisfy

$$0 < \beta_1^+ \leq \beta_2^+ \leq \cdots \leq 1/2 \leq \cdots \leq \beta_2^- \leq \beta_1^{-1} < 1$$

or in other words

$$-1/2 < \lambda_1^- \leq \lambda_2^- \leq \cdots \leq 0 \leq \cdots \leq \lambda_2^+ \leq \lambda_1^+ < 1$$

Inclusions with $\mathcal{C}^{1+\alpha}$ boundaries

Step 1:
Let $\varepsilon > 0$ small enough with respect to the sizes of D_1, D_2
and $w^+ = w_{|\Gamma_1 \cap B(0,\varepsilon)}$ $\quad w^- = w_{|\Gamma_2 \cap B(0,\varepsilon)}$

Theorem

the quantity

$$[w]_{1/2} \quad = \quad \left(\sum_{j=1,2} ||w||^2_{\dot{H}^{1/2}(\Gamma_j)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x) - w^-(x)|^2}{\psi_1(x) + \psi_2(x) + \delta} \right)^{1/2}$$

defines a norm, which is equivalent to the norm $\left(\int_{\mathbb{R}^2} |\nabla u|^2 \right)^{1/2}$
on $\mathcal{H}_S$ *uniformly in* δ*.*

Recall the variational principle for the eigenvalues of $K^{*,\delta}$

$$\beta_n^{\delta,+} \quad = \quad \min_{\substack{F_n \subset \mathcal{H}_S \\ \dim(F_n) = n}} \quad \max_{u \in F_n \setminus \{0\}} \frac{\int_D |\nabla u|^2}{\int_{\mathbb{R}^2} |\nabla u|^2}$$

We obtain estimates

$$\frac{1}{C}\, b_n^{\delta,+} \;\le\; \beta_n^{\delta,+} \;\le\; C\, b_n^{\delta,+} \qquad \forall\, n \ge 1,$$

with a constant C independent of δ, where

$$b_n^{\delta,+} \quad := \quad \min_{\substack{F_n \subset \dot{H}^{1/2} \\ \dim(F_n) = n}} \quad \max_{w \in F_n \setminus \{0\}} \frac{\sum_{j=1}^{2} \|w\|^2_{\dot{H}^{1/2}(\partial D_j)}}{\sum_{j=1}^{2} \|w\|^2_{\dot{H}^{1/2}(\partial D_j)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x) - w^-(x)|^2}{|x|^m + \delta}\, dx}$$

Step 2: Comparison with disks

Let $X_1 = (0, 1 + \delta^{\frac{2}{m}}/2)$, $X_2 = (0, -1 - \delta^{\frac{2}{m}}/2)$, $B_1 = B_1(X_1)$, $B_2 = B_1(X_2)$

Let $D_1^0 = D_1 - \delta/2 e_2$, $D_2^0 = D_2 + \delta/2 e_2$

Let $\Pi : D_1^0 \longrightarrow D_2^0$ be a diffeomorphism such that

$$\Pi(x_1, \psi(x_1)) = (x_1, -\psi_2(x_1)), \quad \text{for } |x_1| \leq \varepsilon$$

Let $G : \partial B_1(0,1) \longrightarrow \partial D_1^0$ be a diffeomorphism such that $G(0,0) = (0,0)$, and consider

$$G_1 = \tau(\frac{\delta}{2}) \circ G \circ \tau(-\frac{\delta^{\frac{2}{m}}}{2})$$

$$G_2 = \tau(-\frac{\delta}{2}) \circ \Pi \circ G \circ \tau(-\frac{\delta^{\frac{2}{m}}}{2}) \circ \Theta$$

where $\tau(\alpha)$ is the translation by αe_2 and Θ the symmetry wrt the x_1- axis.

Then

$$G_1 \;: \partial B_1 \longrightarrow \partial D_1$$

$$G_2 \;: \partial B_2 \longrightarrow \partial D_2$$

and for $|x_1| < \varepsilon$,

$$G_1^{-1}(x_1, \psi(x_1) + \delta/2) \quad = \quad G_2^{-1}(x_1, -\psi_2(x_1) - \delta/2)$$

This parametrization of $\partial D_1, \partial D_2$ leaves the $H^{1/2}$ norm essentially unchanged : for some C independent of δ

$$\frac{1}{C}||v||_{\dot{H}^{1/2}(\partial D_1)} \;\leq\; ||v \circ G_1||_{\dot{H}^{1/2}(\partial B_1(0,1+\frac{\delta^{2/m}}{2}))} \;\leq\; C||v||_{\dot{H}^{1/2}(\partial D_1)}$$

Noticing that $\dfrac{1}{|x|^m + \delta} \leq \dfrac{2\delta^{2/m-1}}{|x|^2 + \delta^{2/m}}$ shows that

$$
\begin{aligned}
b_n^+(D_1, D_2) &\sim \frac{\sum_{j=1}^2 \|w\|^2_{\dot{H}^{1/2}(\partial D_j)}}{\sum_{j=1}^2 \|w\|^2_{\dot{H}^{1/2}(\partial D_j)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x)-w^-(x)|^2}{|x|^m+\delta} dx} \\
&\geq \frac{C\delta^{1-\frac{2}{m}} \sum_{j=1}^2 \|w\|^2_{\dot{H}^{1/2}(\partial D_j)}}{\sum_{j=1}^2 \|w\|^2_{\dot{H}^{1/2}(\partial D_j)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x)-w^-(x)|^2}{|x|^2+\delta^{\frac{2}{m}}} dx} \\
&\geq \frac{C\delta^{1-\frac{2}{m}} \sum_{j=1}^2 \|w \circ G_j\|^2_{\dot{H}^{1/2}(\partial D_j)}}{\sum_{j=1}^2 \|w \circ G_j\|^2_{\dot{H}^{1/2}(\partial D_j)} + \int_{-\varepsilon}^{\varepsilon} \frac{\left|w\circ G_j^+(x)-w\circ G_j^-(x)\right|^2}{|x|^2+\delta^{\frac{2}{m}}} dx} \\
&\sim C\delta^{1-\frac{2}{m}} \, b_n^+(B_1, B_2) \sim C\delta^{1-\frac{1}{m}}
\end{aligned}
$$

Theorem

- D_1, D_2 *are convex, with smooth boundary, and meet at* $X = 0$ *tangentially to the* $x-$*axis when* $\delta = 0$
- *their boundary are parametrized by curves* $(x, \psi_j(x))$ *around* $X = 0$
  with $\psi_1(x) + \psi_2(x) \sim C|x|^m$ *for some* $m \geq 2$

Then the non-degenerate eigenvalues of the Neumann-Poincaré operator form 2 families

$$\begin{aligned} \lambda_{n,+}^{\delta} &= 1/2 - c_{n,+}^{\delta}\, \delta^{\frac{m-1}{m}} + o(\delta^{\frac{m-1}{m}}) \\ \lambda_{n,-}^{\delta} &= -1/2 + c_{n,-}^{\delta}\, \delta^{\frac{m-1}{m}} + o(\delta^{\frac{m-1}{m}}) \end{aligned}$$

where the $c_{n,\pm}^{\delta}$ *form an increasing sequence of positive numbers, that only depend on the shapes of the inclusions and satisfy* $c_{n,\pm}^{\delta} \sim n$ *as* $n \to \infty$

Numerical illustration

Pick $R > 2$ large enough, so that $D_1 \cup D_2 \subset B_{R/2}$ and let $B_\delta : \ H_0^1(B_R) \to H_0^1(B_R)$ defined by

$$\forall\ v \in H_0^1(B_R), \int_{B_R} \nabla B_\delta u(X) \cdot \nabla v(X) \ = \ \int_{D_1 \cup D_2} \nabla u(X) \cdot \nabla v(X).$$

B_δ is self adjoint and of Fredholm type, thus has a spectral decomposition. We denote $b_n^{\delta,\pm}$ its eigenvalues.

Theorem (BTT)

Let $n \geq 1$. There exists a constant C independent of δ and n such that

$$\frac{1}{C} b_n^{\delta,+} \ \leq \ \beta_n^{\delta,+} \ \leq \ C b_n^{\delta,+}.$$

Numerical illustration

We estimate numerically the rate of convergence to 0 of the first non-degenerate eigenvalue $b_1^{\delta,+}$.

To this end, we use the min-max principle to approximate $b_1^{\delta,+}$ by

$$b_{1,N}^{\delta,+} = \min_{u \in V_N} \frac{\int_{D_1 \cup D_2} |\nabla u(X)|^2 dX}{\int_{B_R} |\nabla u(X)|^2 dX}$$

where V_N is a finite dimensional subspace of $H_0^1(B_R)$.

Let $X_1 = (x_1 + iy_1) \in D_1, X_2 = (x_2 + iy_2) \in D_2$ and $n \in \mathbb{N}$. Define $\phi_{n,1}^{\pm}, \phi_{n,2}^{\pm} : \mathbb{R}^2 \longrightarrow \mathbb{C}$ by $\phi_{n,1}(z) = (z - X_1)^n, \phi_{n,2}(z) = (z - X_2)^n$, where $z = x + iy$. Let $w_m, m \geq 1$ be the $H_0^1(D)$ functions which are harmonic in $B_R \setminus \overline{D}$ and such that

$$\begin{cases} w_{4n-3} = Re(\phi_{n,1}) & \text{in } D_1 \\ w_{4n-3} = 0 & \text{in } D_2, \end{cases} \qquad \begin{cases} w_{4n-2} = Im(\phi_{n,1}) & \text{in } D_1 \\ w_{4n-2} = 0 & \text{in } D_2, \end{cases}$$

$$\begin{cases} w_{4n-1} = 0 & \text{in } D_1 \\ w_{4n-1} = Re(\phi_{n,2}) & \text{in } D_2, \end{cases} \qquad \begin{cases} w_{4n} = 0 & \text{in } D_1 \\ w_{4n} = Im(\phi_{n,2}) & \text{in } D_2. \end{cases}$$

Numerical illustration

We consider a conformal triangulation $\mathcal{T}$ of B_R, which is refined in the neck between the 2 inclusions.

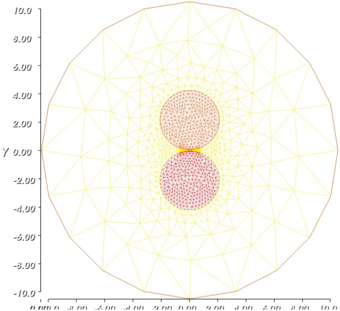


Figure: Mesh for $\delta = 1/16$.

Numerical illustration

Let $\hat{w}_m, m \geq 1$ denote the H^1 projection of w_m on the space of functions which are piecewise linear on $\mathcal{T}$.

We define V_N as the vector space generated by the functions $\hat{w}_m, m \leq 4N$.

We note that the functions $w_m, m \geq 1$ are linearly independent. Together with the functions $w_{0,1}, w_{0,2}$ in $H^1_0(B_R)$ defined by $\Delta w_{0,i} = 0$ in $B_R \setminus \overline{D}$, and

$$\left\{ \begin{array}{lcll} w_{0,1} & = & 1 & \text{in } D_1 \\ w_{0,1} & = & 0 & \text{in } D_2, \end{array} \right. \qquad \left\{ \begin{array}{lcll} w_{0,2} & = & 0 & \text{in } D_1 \\ w_{0,2} & = & 1 & \text{in } D_2, \end{array} \right.$$

they from a basis of $H^1_0(B_R)$.

Numerical illustration

To compute the eigenvalues $b_{1,N}^{\delta,+}$, we form the matrices A and B with entries

$$A_{i,j} = \int_{D_1 \cup D_2} \nabla \hat{w}_i \cdot \nabla \hat{w}_j, \qquad B_{i,j} = \int_{B_R} \nabla \hat{w}_i \cdot \nabla \hat{w}_j,$$

and then compute the generalized eigenvalues of the system $AU = \beta BU$.

We have used the software Freefem++ to compute the vectors $\hat{w}_m$, and Scilab to solve the above matrix eigenvalue problem.

Numerical illustration

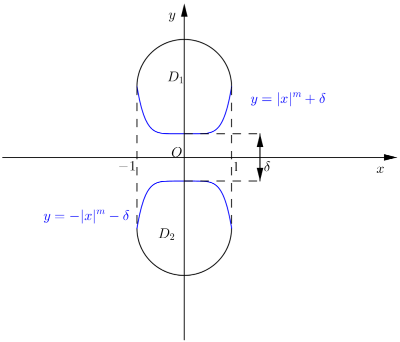


Figure: Domains D_1 and D_2

The points X_1 and X_2 in the construction of the space V_N, are the centers of the perturbed discs.

Numerical illustration

We deduce from the asymptotic analysis that

$$\log b_{1,N}^{\delta,+} \sim \log c_1^+ + \frac{m-1}{m} \log \delta$$

as δ tends to 0.

The following table provide the numerical results for δ between $1/2$ and $1/2^7$, and $N = 39$.

m	Equation of the line	Theory	Error
$m=2$	$t = -0.7934156 + 0.4307516s$	$\frac{1}{2} = 0.5$	0.0692484
$m=6$	$t = -0.1401772 + 0.8003479s$	$\frac{5}{6} \simeq 0.83$	0.03298543
$m=9$	$t = -0.2357561 + 0.8508496s$	$\frac{8}{9} \simeq 0.89$	0.03803929

We remark that the computed slopes are in a good agreement with the expected theoretical values.

Numerical illustration

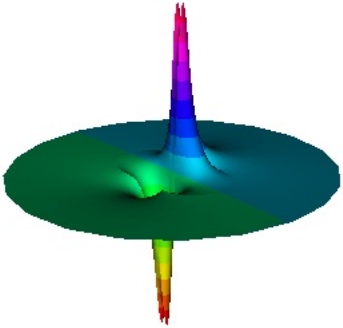


Figure: eigenfunction for m=9

Conclusion

• We obtain the asymptotic of the eigenvalues of $K^{*,\delta}$ as $\delta \to 0$ which may be used for design purposes in optimizing plasmon resonances

• In the case of disks, one can read off the blow up rate of ∇u_δ on the eigenvalues of the Neumann-Poincaré operator

• How do we derive the pointwise convergence of the spectral decomposition?

• In the case of disks, [Kang-Lim-Yun] obtained an asymptotic expansion of the solution, isolating a dipole-like singular part. It would be nice to extend this to more general geometries of contact

• Using the same approach can we derive estimates of the resonances and the gradients in the 3d case?

• Inverse problems related to surface plasmonic resonances (Biosensing) (Phd thesis of A. Banani).

• Asymptotic expansion of plasmonic resonances of periodic distributed nano-particles, nano-holes (Phd thesis of L. Salesses)+[Bonnetier-Dapogny-Triki 16].