Elliptic estimates in composite media with smooth inclusions: an integral equation approach

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Abstract

We consider a scalar elliptic equation for a composite medium consisting of homogeneous $\mathcal{C}^{1,\alpha_0}$ inclusions, $0 < \alpha_0 \leq 1$, embedded in a constant matrix phase. When the inclusions are separated and are separated from the boundary, the solution has an integral representation, in terms of potential functions defined on the boundary of each inclusion. We study the system of integral equations satisfied by these potential functions as the distance between two inclusions tends to 0. We show that the potential functions converge in $\mathcal{C}^{0,\alpha}$, $0 < \alpha < \alpha_0$ to limiting potential functions, with which one can represent the solution when the inclusions are touching. As a consequence, we obtain uniform $\mathcal{C}^{1,\alpha}$ bounds on the solution, which are independent of the inter–inclusion distances.

1 Introduction

In a bounded domain $\Omega \subset I\!R^2$, we consider a composite medium consisting of a finite number of inclusions embedded in a matrix phase. We assume that the inclusions and the matrix have (different) constant, scalar conductivities. The resulting, spatially varying, piecewise constant conductivity is denoted by $a(\cdot)$. Given a current g on the boundary $\partial\Omega$, with $\int_{\partial\Omega} g \, d\sigma = 0$, we consider the solution u to the elliptic equation

$$\nabla \cdot (a(\cdot)\nabla u) = 0 \quad \text{in } \Omega, \quad \text{with} \quad a(\cdot)\partial_\nu u = g \quad \text{on } \partial\Omega \ ,$$

in other words, we consider the continuous function u, which is harmonic in each inclusion as well as in the matrix, which satisfies the usual transmission conditions

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across the inclusion boundaries, and which has the prescribed co-normal derivative g on $\partial\Omega$. To make u unique we impose the condition $\int_{\partial\Omega} u \, d\sigma = 0$.
In this paper, we are interested in apriori estimates for the solution u, and in particular its gradient. We assume that Ω has a smooth boundary, and that the imposed current is smooth. When the inclusions are merely Lipschitz, it is well known (from elliptic theory in domains with corners) that ∇u is generally not uniformly bounded, *i.e.*, generally not in L^∞. On the other hand, when the inclusions are smooth (say $\mathcal{C}^{1,\alpha_0}, 0 < \alpha_0 \leq 1$) and when they are not mutually touching and do not touch $\partial\Omega$, it is equally well known that ∇u is bounded. A natural question is whether ∇u stays uniformly bounded, even as some of the inclusions get close.
This question has been addressed in several papers (see for example [11] and [20]). It has been established that ∇u is bounded in $L^\infty(\Omega)$ independly of the distance between the smooth inclusions. The answer given in [20] is actually quite a bit more general. It addresses the case of a divergence form elliptic equation with 'piecewise Hölder coefficients': assume there exist numbers $0 < \alpha_0, \mu \leq 1$, $0 < c_0 < C_0$, and a positive integer M such that

i. Ω contains M possibly touching inclusions D_l, $1 \leq l \leq M$, each of which is a $\mathcal{C}^{1,\alpha_0}$ subdomain.

ii. For any $1 \leq l \leq M$, $\mathrm{dist}(\overline{D_l}, \partial\Omega) > 0$,

iii. In each inclusion, and in the remaining part $D_{M+1} := \Omega \setminus \cup_{1\leq l\leq M}\overline{D_l}$, the conductivity satisfies $c_0 < a_{|\overline{D}_l} < C_0$, and has $\mathcal{C}^\mu$ regularity.

Then

$$\sum_{l=1}^{M+1} ||u||_{\mathcal{C}^{1,\alpha}(\overline{D}_l\cap\Omega_\varepsilon)} \;\leq\; C||g||_{L^2(\partial\Omega)}, \quad \text{for any } 0 < \alpha < \min\{\mu, \frac{\alpha_0}{2(\alpha_0+1)}\}, \tag{1}$$

where Ω_ε, $\varepsilon > 0$, denotes the set

$$\Omega_\varepsilon \;=\; \{X \in \Omega,\ \mathrm{dist}(X, \partial\Omega) > \varepsilon\}.$$

The constant C depends on $\varepsilon, \alpha, M, c_0, C_0, \mu, \Omega$ and the appropriate $\mathcal{C}^{1,\alpha}$ "norms" of the parametrizations of the inclusion boundaries. But note that C is independent of the inter–inclusion distance. The proof given in [20] uses elliptic blow-up techniques and maximum principles, and is thus restricted to scalar problems.
In a subsequent paper [19], Y.-Y. Li and L. Nirenberg extended the above result to strongly elliptic systems, with the same restriction $0 < \alpha < \min\{\mu, \frac{\alpha_0}{2(\alpha_0+1)}\}$ for the regularity "measure" of u. In the scalar case, the recent results of [13] prove that the restriction can be lifted to $\alpha \leq \min\{\mu, \alpha_0\}$.

In the case of perfectly conducting or perfectly insulating inclusions the gradients may blow up as the inter-inclusion distance, δ, approaches 0. In [7], the solution for perfectly conducting inclusions is shown to satisfy

$$\left\{ \begin{array}{lcll} ||\nabla u||_{L^\infty} & \leq & \frac{C}{\sqrt{\delta}}||u||_{L^2(\partial\Omega)} & \text{for } n=2, \\ ||\nabla u||_{L^\infty} & \leq & \frac{C}{\delta|\ln\delta|}||u||_{L^2(\partial\Omega)} & \text{for } n=3, \\ ||\nabla u||_{L^\infty} & \leq & \frac{C}{\delta}||u||_{L^2(\partial\Omega)} & \text{for } n=4, \end{array} \right. \tag{2}$$

where n is the ambient dimension. The case $n = 2$ was derived independently by Yun, using conformal mapping techniques [22]. The picture is less complete for the case of insulating inclusions, see [8].

For $n = 2$ and for circular inclusions, one can obtain very precise bounds in terms of both contrast and inter-inclusion distance, since the solution has a series representation that lends itself to asymptotic analysis [3, 4, 12, 21]. Optimal upper and lower bounds on the potential gradients are derived in [3, 4] for nearly touching pairs of circular inclusions. In the case of 2 disks, a decomposition of the solution into a singular part, and a part that remains uniformly bounded with respect to δ, is given [5].

When the conductivity is piecewise constant, and when the inclusions are $\mathcal{C}^{1,\alpha_0}$, mutually separated and separated from the boundary, then one can represent u in the form

$$u(X) \;=\; \sum_{l=1}^{M} S_l\varphi_l(X) + H(X) \;\;, \tag{3}$$

where H is a harmonic function, where each φ_l is defined on ∂D_l, and where S_l denotes the single layer potential on ∂D_l. Invoking the transmission conditions on ∂D_l and the Neumann condition on $\partial\Omega$, we can derive a system of integral equations, for the φ_l's, and an associated (implicit) formula for H. As each inclusion has $\mathcal{C}^{1,\alpha_0}$ regularity, results from classical potential theory (see, e.g., [15]) show that this system is invertible. Detailed facts about the regularity of u may be deduced from the representation (3).

The aim of this paper is to show that the system of integral equations for the φ_l's is uniformly invertible in $\mathcal{C}^{0,\alpha}$ as inclusions get close. The associated uniform estimates on the inverse can then be used to derive a priori estimates for the solution, u, in $\mathcal{C}^{1,\alpha}$ norms.

The integral representation (3) of solutions has also been used in other related contexts. In particular, recent works have focused on the connection between the bounds on ∇u and the spectral properties of the kernel of the integral equation system (the Neumann–Poincaré operator) for varying coefficient contrast and inter-inclusion distance [1, 10].

For simplicity we always assume that the inclusions are convex, and that any two that asymptotically meet only meet at one point. Since the regularity of u and the corresponding estimates only depend on the geometry of the inclusions locally, we shall restrict ourselves to the case of two inclusions, D_1 and D_2, of size $O(1)$, that asymptotically meet (with a horizontal tangent) at the point 0, see Figure 1. We denote $\Gamma_i := \partial D_i, i = 1, 2$. For simplicity, we assume that the matrix phase has conductivity 1 and that both inclusions have conductivity $k \neq 1$. For $\delta > 0$, we consider the situation

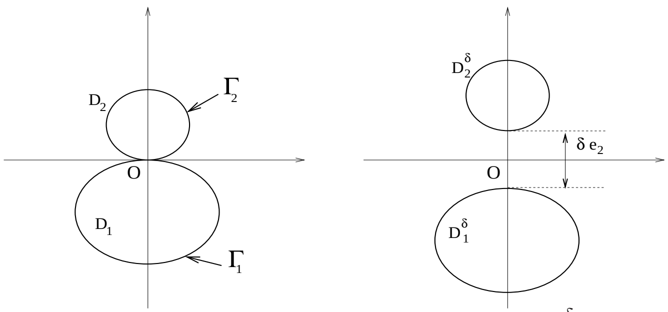

Figure 1: The touching and near–touching configurations. $D_1^\delta = D_1 - \frac{\delta}{2}e_2$, and $D_2^\delta = D_2 + \frac{\delta}{2}e_2$.

where the inclusions are at a distance δ apart, say in the unit vertical direction e_2. As we shall see, the corresponding system of integral equation for the potential functions $(\varphi_1^\delta, \varphi_2^\delta)$ may be written

$$T^\delta \begin{pmatrix} \varphi_1^\delta \\ \varphi_2^\delta \end{pmatrix} := \begin{pmatrix} \lambda I - K_1^* & L_2^\delta \\ L_1^\delta & \lambda I - K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1^\delta \\ \varphi_2^\delta \end{pmatrix} = \begin{pmatrix} g_1^\delta \\ g_2^\delta \end{pmatrix}. \tag{4}$$

Here, $\lambda = \frac{k+1}{2(k-1)}$, (g_1^δ, g_2^δ) are known functions (given in terms of the boundary flux g) K_i^* denotes the trace on Γ_i of the normal derivative of the single layer potential on Γ_i, and L_2^δ denotes the normal derivative on Γ_1 (or rather, on the $-\delta$ vertical translate of Γ_1) of the single layer potential defined on Γ_2. To be precise

$$L_2^\delta \varphi_2^\delta(X) = -\nu_1(X) \cdot \nabla S_2 \varphi_2^\delta(X - \delta e_2), \quad X \in \Gamma_1,$$

and similarly for L_1^δ. We notice that even though the physical situation is as in the right picture of Figure 1, we use potentials φ_i^δ that live on the δ independent curves $\Gamma_i = \partial D_i$, as in the left picture of Figure 1. Thus, vertical translations (by $\pm\delta$ and $\pm\delta/2$) appear in appropriate places. We also notice that the parameter λ always satisfies $|\lambda| > 1/2$.

Throughout the paper, we assume that the inclusions are $\mathcal{C}^{1,\alpha_0}$, with $0 < \alpha_0 \leq 1$ and we seek the potentials in the space $\mathcal{C}^{0,\alpha}$, $0 < \alpha < \alpha_0$, of slightly less regular functions. When $\delta > 0$, the kernel of L_2^δ is smooth, so that L_2^δ is a compact operator from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1)$ [18]. Similarly, L_1^δ is compact, so that by Fredholm theory, the system (4) is invertible in $\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$.

If the operators (L_1^δ, L_2^δ) were convergent in the operator norm ($C^{0,\alpha}$ to $C^{0,\alpha}$) as $\delta \to 0$, and if we could show that the limiting system corresponding to (4) is invertible, then the operators $(T^\delta)^{-1}$ would converge in norm to the inverse of that limiting system. In particular $(T^\delta)^{-1}$ would be norm bounded, and we would immediately obtain uniform piecewise $C^{1,\alpha}$ estimates for u. It is indeed possible to show that the operators (L_1^δ, L_2^δ) converge *pointwise* to some (L_1^0, L_2^0), and that the limiting system corresponding to (4) is necessarily invertible. However, as we shall show that the operators (L_1^0, L_2^0) are not compact, the convergence of (L_1^δ, L_2^δ) cannot take place in the operator norm. Therefore the simple argument above cannot be used to obtain uniform estimates for u. We note here that we are not entirely sure whether this "degenerate" picture is special to dimension two. In our opinion it would be very interesting to resolve this question, and thus to understand any potential "qualitative" difference in the behavior of the gradients near contact points in dimension two versus dimension three and higher.

Due to the lack of norm convergence, mentioned above, we appeal to results about collectively compact operators established by P.M. Anselone [6]. These results require only pointwise convergence and invertibility of the limiting operator to garantee pointwise convergence (and thus uniform norm-boundedness) of the $(T^\delta)^{-1}$'s. It is very useful to note that the limiting operators (L_1^0, L_2^0) are nearly compact: their kernel is singular only at one point, namely where the two inclusions touch.

We use this observation to split the operators T^δ as a sum of operators the supports of which depend on a small parameter ε. Due to our assumptions the curve Γ_1 can, near $X = 0$, be parametrized as $(x, \psi_1(x))$ with $\psi_1 \in \mathcal{C}^{1,\alpha_0}(I\!R)$ and such that $\psi_1(0) = \psi_1'(0) = 0$, and similarly for the curve Γ_2. Given $\varepsilon > 0$, we introduce approximate curves $\psi_{1,\varepsilon}, \psi_{2,\varepsilon}$ which satisfy

$$\left\{ \begin{array}{lcl} \psi_{j,\varepsilon} & \equiv & \psi_j \qquad j = 1, 2, \ |x| \leq \varepsilon \\ ||\psi_{j,\varepsilon}||_{\mathcal{C}^{1,\alpha}} & \leq & 2\,||\psi_j||_{\mathcal{C}^{1,\alpha_0}}\, \varepsilon^\nu \end{array} \right.$$

for any $0 < \alpha < \alpha_0$. where $\nu = \alpha_0 - \alpha > 0$, see Figure 2. We then split L_2^δ as

$$L_2^\delta \ = \ \chi \left(K_2^{\varepsilon,\delta} + \frac{1}{2\pi\sqrt{1 + [\psi_{1,\varepsilon}'(x)]^2}} (J_2^{\varepsilon,\delta} + I_2^{\varepsilon,\delta}) \right) + (1 - \chi) L_2^\delta,$$

where χ is a smooth cut-off function that is identically one near the origin. The term $K_2^{\varepsilon,\delta}$ is (near $X = 0$) the difference between L_2^δ and the normal derivative at the *approximate point* $(x, \psi_{1,\varepsilon}(x) - \delta e_2)$ of the single layer potential on the *approximate curve* $y \to (y, \psi_{2,\varepsilon}(y))$. Since the original and approximate curves coincide in an ε–neighborhood of the origin, the operators $K_2^{\varepsilon,\delta}$ are collectively compact with respect to δ. The term involving $I_2^{\varepsilon,\delta}$ is the normal derivative at the *approximate point* $(x, \psi_{1,\varepsilon}(x) -$

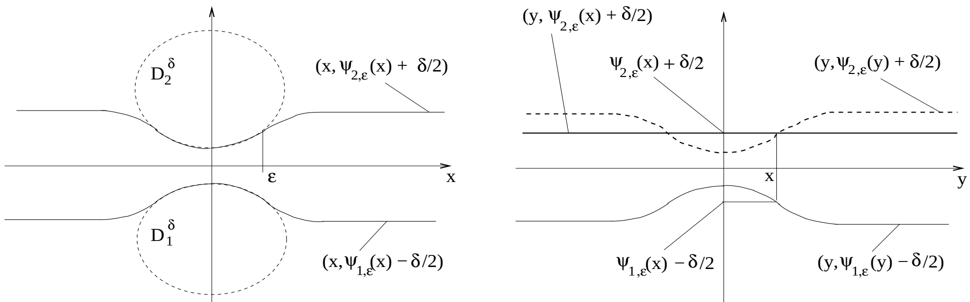

Figure 2: The approximate curves introduced in the splitting of L_2^δ.

δe_2) of the single layer potential on the *straight line* $y \to (y, \psi_{2,\varepsilon}(x))$, and the term involving $J_2^{\varepsilon,\delta}$ is the remainder, see Figure 2.

We decompose L_1^δ likewise and define

$$\begin{aligned}
\Lambda_{\varepsilon,\delta} &= \begin{pmatrix} \lambda I & \frac{\chi(X)}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}(x)]^2}}(J_2^{\varepsilon,\delta} + I_2^{\varepsilon,\delta}) \\ \frac{\chi(X)}{2\pi\sqrt{1+[\psi'_{2,\varepsilon}(x)]^2}}(J_1^{\varepsilon,\delta} + I_1^{\varepsilon,\delta}) & \lambda I \end{pmatrix} \\
C_{\varepsilon,\delta} &= \begin{pmatrix} -K_1^* & \chi K_2^{\varepsilon,\delta} + (1-\chi)L_2^\delta \\ \chi K_1^{\varepsilon,\delta} + (1-\chi)L_1^\delta & -K_2^* \end{pmatrix},
\end{aligned}$$

so that $T^\delta = \Lambda_{\varepsilon,\delta} + C_{\varepsilon,\delta}$. In this decomposition, the operators $C_{\varepsilon,\delta}$ are collectively compact, whereas the operators $\Lambda_{\varepsilon,\delta}$ are pointwise convergent and invertible, with uniformly norm-bounded inverses. Since they do incorporate a term from the L_i^δ, the operators $\Lambda_{\varepsilon,\delta}$ are, however, not "diagonal".

Given $|\lambda| > 1/2$, we show in Lemmas 6 and 7, that we can fix $\varepsilon > 0$ small enough, so that the norm of the off–diagonal terms of $\Lambda_{\varepsilon,\delta}$ is strictly smaller than $|\lambda|$, uniformly with respect to $0 < \delta < 1$. The operators $\Lambda_{\varepsilon,\delta}, 0 < \delta < 1$ are thus invertible in $\mathcal{C}^{0,\alpha}(\Gamma_1)\times \mathcal{C}^{0,\alpha}(\Gamma_2)$. We then show (Lemmas 5 and 8) that $(\Lambda_{\varepsilon,\delta}, C_{\varepsilon,\delta})$ converge pointwise to some limiting operators $(\Lambda_{\varepsilon,0}, C_{\varepsilon,0})$, as $\delta \to 0$ (for $C_{\varepsilon,\delta}$ in a collectively compact fashion). These limiting operators correspond to an integral formulation of the limiting elliptic problem with $\delta = 0$. As a consequence, we obtain our main result, Theorem 1: the operators T^δ are invertible operators in $\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2))$, and their inverses are bounded independently of δ. Moreover, the operators $(T^\delta)^{-1}$ converge pointwise to $(T^0)^{-1}$ as $\delta \to 0$.

The paper is organized as follows: In Section 2, we make precise our assumptions on the geometry, we describe in detail the system of integral equations, when the inclusions are not touching, and we also derive the splitting of the system as briefly explained above. Our main result is found and proven in Section 3. The proof depends on a number of technical lemmas that are precisely stated in this section, but the verifications of which are relegated to appendices A-D. Appendices A-C are devoted to proving Lemmas 5–7, that concern the properties of the operators $(K_2^{\varepsilon,\delta}, J_2^{\varepsilon,\delta}, I_2^{\varepsilon,\delta})$, for ε sufficiently small. Appendix D gives a proof of Lemma 8 which asserts that, for fixed $\varepsilon > 0$, the aforementioned operators converge pointwise when $\delta \to 0$. Finally, Appendix E is devoted to a proof of the non–compactness of the limiting operators (L_1^0, L_2^0). Although this result is not needed for the proof of our main Theorem, we feel its inclusion is nonetheless relevant, since it was what motivated a significant part of our analysis.

2 Layer potentials for a system of 2 inclusions

2.1 Notations and assumptions

We recall that a closed curve $\Gamma \subset I\!R^2$ has regularity $C^{1,\alpha}$ if it can be covered by a local set of charts

$$\psi_j \ : x \in I_j \quad \longrightarrow \quad (\psi_{j,1}(x), \psi_{j,2}(x)) \subset I\!R^2 \ ,$$

where I_j, $1 \le j \le J$, are open intervals of $I\!R$, and where the $\psi_{j,i}$'s are $C^{1,\alpha}(\overline{I_j})$ functions with $(\psi'_{j,1})^2 + (\psi'_{j,2})^2 > 0$. We say that a continuous function f is of regularity $\mathcal{C}^{0,\alpha}(\Gamma)$ if for any of the local charts

$$|f \circ \psi_j|_{\mathcal{C}^{0,\alpha}(\overline{I_j})} \quad := \quad \sup_{x,\hat{x} \in \overline{I_j},\, |x-\hat{x}|<1} \frac{|f(\psi_{j,1}(x), \psi_{j,2}(x)) - f(\psi_{j,1}(\hat{x}), \psi_{j,2}(\hat{x}))|}{|x-\hat{x}|^\alpha} \ \le \ C.$$

The norm on $\mathcal{C}^{0,\alpha}(\Gamma)$ is defined by

$$||f||_{\mathcal{C}^{0,\alpha}(\Gamma)} \quad = \quad \max\left(||f||_{L^\infty(\Gamma)}, \max_{1\le j \le J} |f \circ \psi_j|_{\mathcal{C}^{0,\alpha}(\overline{I_j})} \right) \quad .$$

We consider a bounded smooth domain $\Omega \subset I\!R^2$ containing 0. D_1 and D_2 are two touching, simply connected domains (inclusions) contained in Ω; their boundaries are denoted Γ_1 and Γ_2. We assume that D_1 lies in the lower half–plane $x_2 < 0$, D_2 in the upper half–plane, and make the following assumptions about the geometry:

A1. The inclusions are strictly convex and only meet at the point 0.

A2. Around the point 0, Γ_1 and Γ_2 are parametrized by 2 curves $(x, \psi_1(x))$ and $(x, \psi_2(x))$ respectively. The graph of ψ_1 (resp. ψ_2) lies below (resp. above) the x–axis.

A3. The inclusions D_1 and D_2 are globally $\mathcal{C}^{1,\alpha_0}$, for some $0 < \alpha_0 \leq 1$. In particular, each function ψ_j has regularity $\mathcal{C}^{1,\alpha_0}$.

A4. D_1 and D_2 lie strictly inside Ω, i.e., $\mathrm{dist}(\partial\Omega, \overline{D_1 \cup D_2}) \; > \; c_0$ for some $c_0 > 0$.

Throughout the text, C is a generic positive constant, that may only depend on the geometry of each inclusion, but not on the parameters δ, ε_0 and ε introduced below.

2.2 The system of integral equations

Let $g \in \mathcal{C}^\infty(\partial\Omega)$, such that $\int_{\partial\Omega} g = 0$. We first introduce the diffusion equation

$$\left\{ \begin{array}{rcll} \mathrm{div}(a_0(x)\nabla u_0) & = & 0 & \text{in } \Omega, \\ \partial_\nu u_0(x) & = & g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_0 & = & 0, & \end{array} \right. \tag{5}$$

where the conductivity a_0 is equal to $k > 0, k \neq 1$, in $D_1 \cup D_2$, and to 1 in $\Omega \backslash (D_1 \cup D_2)$.

The real physical situation we are interested in is one in which the two inclusions are separated by a small distance: For $\delta > 0$, we set $D_1^\delta = D_1 - \delta/2 e_2$, $D_2^\delta = D_2 + \delta/2 e_2$, and we denote by a_δ the corresponding conductivity distribution. Let u_δ be the solution to

$$\left\{ \begin{array}{rcll} \mathrm{div}(a_\delta(x)\nabla u_\delta) & = & 0 & \text{in } \Omega, \\ \partial_\nu u_\delta(x) & = & g & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_\delta & = & 0. & \end{array} \right. \tag{6}$$

In other words, the function u_δ is harmonic inside and outside the inclusions D_1^δ, D_2^δ, and satisfies the transmission conditions

$$u_\delta^+ \; = \; u_\delta^- \qquad \frac{\partial u_\delta^+}{\partial \nu} \; = \; k \frac{\partial u_\delta^-}{\partial \nu}, \qquad \text{on } \partial D_i^\delta. \tag{7}$$

Here u_δ^+ (resp. u_δ^-) denotes the solution outside (resp. inside) the inclusions, and ν is the outside normal to ∂D_i^δ. Since the coefficients $a_\delta = 1 + (k-1)\chi_{D_1^\delta \cup D_2^\delta}$ converge to a in $L^p(\Omega)$ for any $p < \infty$, it follows from Meyers' theorem [9] that

$$\lim_{\delta \to 0} ||u_\delta - u_0||_{H^1(\Omega)} \; = \; 0. \tag{8}$$

Let $G(X,Y) = \frac{1}{2\pi} \ln(|X - Y|)$ denote the fundamental solution to the Laplace operator in dimension 2. Let $S_{\partial\Omega}$ and $D_{\partial\Omega}$ denote the single and double layer potentials on $\partial\Omega$, defined on $L^2(\partial\Omega)$ by

$$\begin{aligned} S_{\partial\Omega}f(X) &= \int_{\partial\Omega} G(X,Y)f(Y)d\sigma_Y \quad X \in I\!R^2 \setminus \partial\Omega, \\ D_{\partial\Omega}f(X) &= \int_{\partial\Omega} \partial_{\nu_Y} G(X,Y)f(Y)d\sigma_Y \quad X \in I\!R^2 \setminus \partial\Omega, \end{aligned}$$

and let S_i denote the single layer potential on Γ_i, defined on $L^2(\Gamma_i)$ by

$$S_i f(X) = \int_{\Gamma_i} G(X,Y)f(Y)d\sigma_Y \quad X \in I\!R^2 \setminus \Gamma_i.$$

We introduce the harmonic parts of u_0 and u_δ (see [2] sect. 1.4)

$$\left\{ \begin{array}{rcll} H_0(X) &=& -S_{\partial\Omega}g(X) + D_{\partial\Omega}(u_{0|\partial\Omega})(X) & X \in \Omega \\ H_\delta(X) &=& -S_{\partial\Omega}g(X) + D_{\partial\Omega}(u_{\delta|\partial\Omega})(X) & X \in \Omega. \end{array} \right. \tag{9}$$

Lemma 1 *Let $\delta_0 > 0$, and $\omega \subset\subset \Omega$, such that $D_1^\delta \cup D_2^\delta \subset \omega$, for $\delta < \delta_0$. Then, for all $n \in \mathbf{N}$, there exists $C = C(n, k, \Omega, dist(\partial\Omega, \omega) > 0$, such that*

$$\forall\, \delta < \delta_0, \quad ||H_\delta||_{\mathcal{C}^n(\overline{\omega})} \leq C||g||_{L^2(\partial\Omega)}. \tag{10}$$

We furthermore have that

$$\lim_{\delta \to 0} ||H_\delta - H_0||_{\mathcal{C}^n(\overline{\omega})} = 0. \tag{11}$$

Proof: The definition of H_δ and H_0 immediately gives

$$H_\delta - H_0 = D_{\partial\Omega}(u_{\delta/\partial\Omega}) - D_{\partial\Omega}(u_{0/\partial\Omega}) \ ,$$

and since ω is strictly inside Ω we may estimate

$$||H_\delta - H_0||_{\mathcal{C}^n(\overline{\omega})} \leq C||u_\delta - u_0||_{L^2(\partial\Omega)} \leq C||u_\delta - u_0||_{H^1(\Omega)} \ ,$$

where the constants C only depend on n, Ω and dist$(\partial\Omega, \omega)$. The assertion (8) now leads to the desired convergence (11). To prove the uniform estimate (10), we see that

$$||H_\delta||_{\mathcal{C}^n(\overline{\omega})} \leq C\left(||g||_{L^2(\partial\Omega)} + ||u_\delta||_{L^2(\partial\Omega)}\right) \ , \tag{12}$$

and

$$||u_\delta||_{L^2(\partial\Omega)} \leq C||u_\delta||_{H^1(\Omega)} \leq C||g||_{L^2(\partial\Omega)} \ , \tag{13}$$

where the constants C only depend on n, k Ω and dist$(\partial\Omega, \omega)$. For the last estimate we have used the Trace Theorem as well as an elliptic energy estimate. A combination of (12) and (13) gives the desired estimate for H_δ. ∎

Let $\delta > 0$. We define for $X \in \Gamma_1$

$$\varphi_1^\delta(X) \;=\; \left(\partial_\nu u_\delta^+ - \partial_\nu u_\delta^-\right)_{|\partial D_1^\delta}(X - \frac{\delta}{2}e_2)$$

and for $X \in \Gamma_2$

$$\varphi_2^\delta(X) \;=\; \left(\partial_\nu u_\delta^+ - \partial_\nu u_\delta^-\right)_{|\partial D_2^\delta}(X + \frac{\delta}{2}e_2).$$

By repeated integrations by parts, it is easy to calculate that u_δ can be represented as

$$u_\delta(X) \;=\; S_1\varphi_1^\delta(X + \frac{\delta}{2}e_2) \;+\; S_2\varphi_2^\delta(X - \frac{\delta}{2}e_2) \;+\; H_\delta(X). \tag{14}$$

The standard jump relations for a single layer potential also show that the functions φ_1^δ and φ_2^δ solve the following system of integral equations

$$\begin{cases} (\lambda I - K_1^*)\varphi_1^\delta(X) \;-\; \frac{\partial}{\partial\nu}S_2\varphi_2^\delta(X - \delta e_2) \;=\; \partial_\nu H_\delta(X - \frac{\delta}{2}e_2) \quad X \in \Gamma_1 \\ \\ -\frac{\partial}{\partial\nu}S_1\varphi_1^\delta(X + \delta e_2) \;+\; (\lambda I - K_2^*)\varphi_2^\delta(X) \;=\; \partial_\nu H_\delta(X + \frac{\delta}{2}e_2) \quad X \in \Gamma_2. \end{cases} \tag{15}$$

In this system, $\lambda = \frac{k+1}{2(k-1)} \in I\!R \setminus [-1/2, 1/2]$, and K_i^* denotes the operator defined on $L^2(\Gamma_i)$ by

$$K_i^* f(X) \;=\; \frac{1}{2\pi}\int_{\Gamma_i} \frac{(X-Y)\cdot\nu(X)}{|X-Y|^2} f(Y)\, ds_Y.$$

Classical results from potential theory show that for any $0 < \alpha < \alpha' < \alpha_0$,

$$||S_i(\varphi_i^\delta)||_{C^{1,\alpha}(\overline{D_i^\delta})} + ||S_i(\varphi_i^\delta)||_{C^{1,\alpha}(\Omega\setminus D_i^\delta)} \leq C||\varphi_i^\delta||_{C^{0,\alpha'}(\Gamma_i)} \quad , \tag{16}$$

for $i = 1, 2$, see [18]. Based on the representation formula (14) and Lemma 1 we thus immediately get the following result.

Lemma 2 *Let u_δ be the solution to (6), and let $(\varphi_1^\delta, \varphi_2^\delta)$ be the solution to (15), where H_δ is given by (9). For any small $\eta > 0$, let Ω_η denote the set $\Omega_\eta = \{x \in \Omega, dist(x, \partial\Omega) > \eta\}$. Then for any $0 < \alpha < \alpha' < \alpha_0$,*

$$\begin{aligned} ||u_\delta||_{C^{1,\alpha}(\overline{D_1^\delta})} + ||u_\delta||_{C^{1,\alpha}(\overline{D_2^\delta})} + ||u_\delta||_{C^{1,\alpha}(\Omega_\eta\setminus(D_1^\delta \cup D_2^\delta))} \\ \leq C\left(\sum_{i=1}^{2} ||\varphi_i^\delta||_{C^{0,\alpha'}(\Gamma_i)} + ||g||_{L^2(\partial\Omega)}\right) \quad , \end{aligned}$$

for some constant C, depending on $\alpha, \alpha', \alpha_0, \Omega, k, \eta$, but independent of δ.

According to this lemma we obtain the desired piecewise Hölder estimates (1) on ∇u_δ, if we can establish uniform $C^{0,\alpha}, \alpha < \alpha_0$, bounds on the potentials φ_i^δ. Since H_δ is bounded uniformly in any norm on the curves Γ_i (by Lemma 1) such uniform bounds on the φ_i^δ follow if we can verify that the operator on the left-hand side of (15) has a uniformly bounded inverse as an operator on $C^{0,\alpha}(\Gamma_1) \times C^{0,\alpha}(\Gamma_2), \alpha < \alpha_0$. This verification is the focus of the remainder of this paper.

2.3 Decomposition of the system of integral equations

In this section, we begin our detailed study of the system of integral equations

$$T^\delta \begin{pmatrix} \varphi_1^\delta \\ \varphi_2^\delta \end{pmatrix} := \begin{pmatrix} \lambda I - K_1^* & L_2^\delta \\ L_1^\delta & \lambda I - K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1^\delta \\ \varphi_2^\delta \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{17}$$

where, for $(\varphi_1, \varphi_2) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$,

$$\left\{ \begin{array}{lcll} L_2^\delta \varphi_2(X) & = & -\frac{\partial}{\partial \nu} S_2 \varphi_2(X - \delta e_2) & \quad X \in \Gamma_1 \\ & & & \\ L_1^\delta \varphi_1(X) & = & -\frac{\partial}{\partial \nu} S_1 \varphi_1(X + \delta e_2) & \quad X \in \Gamma_2, \end{array} \right. \tag{18}$$

When $\delta > 0$, classical potential theory applies, and one finds that T^δ is a continuous linear mapping on $\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$, invertible with bounded inverse, for any $0 < \alpha < \alpha_0$, and for any $|\lambda| > 1/2$.
Our goal is to study the behavior of T^δ and its inverse as $\delta \to 0$. As the inclusions come to touch, the terms $\partial_\nu S_2 \varphi_2$ and $\partial_\nu S_1 \varphi_1$ may become singular at the contact point. To isolate this difficulty, we decompose T^δ as a sum $\Lambda_{\varepsilon,\delta} + C_{\varepsilon,\delta}$, where for a fixed $\varepsilon > 0$ sufficiently small, the operator $\Lambda_{\varepsilon,\delta}$ contains the singular part of T^δ (i.e., the identity plus a piece of the off-diagonal terms) and where $C_{\varepsilon,\delta}$ is compact.

We fix a small parameter $0 < \varepsilon_0 < 1$ so that

$$\frac{1}{2} < \frac{1+\varepsilon_0}{2} < |\lambda|. \tag{19}$$

Let $R_0 = 2(1 + \varepsilon_0^{-1})$. By a rescaling of Ω, if necessary, we may assume that each inclusion is sufficiently large so that the intersection of $(\Gamma_1 \cup \Gamma_2) \cap B(0, 2R_0)$ with the vertical axis is reduced to the contact point 0. In other words, the 'South pole' of Γ_1 and the 'North pole' of Γ_2 are at a distance greater than $2R_0$ from the contact point. Let χ be a smooth cut-off function, such that

$$\left\{ \begin{array}{l} 0 \le \chi(X) \le 1, \\ \chi(X) = 1 \quad \text{for } X \in B(0, \varepsilon_0), \\ \mathrm{Supp}(\chi) \in B(0, R_0), \\ ||\nabla \chi||_\infty \le \varepsilon_0. \end{array} \right. \tag{20}$$

We also assume that ε_0 is sufficiently small so that around the contact point $X = 0$, the curves Γ_i can be parametrized by

$$\left\{ \begin{array}{lll} |x| \leq \varepsilon_0 & \longrightarrow & X = (x, \psi_1(x)) \in \Gamma_1, \\ |y| \leq \varepsilon_0 & \longrightarrow & Y = (y, \psi_2(y)) \in \Gamma_2. \end{array} \right. \tag{21}$$

Lemma 3 *Given $0 < \varepsilon_0 < 1$ for which (21) holds, and given $0 < \alpha < 1$, there exists an operator $E\ : \mathcal{C}^{0,\alpha}(\Gamma_2) \longrightarrow \mathcal{C}^{0,\alpha}(I\!R)$, such that for any $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$,*

$$\left\{ \begin{array}{lll} ||E\varphi||_{0,\alpha} & \leq & (1+\varepsilon_0)||\varphi||_{0,\alpha} \\ E\varphi(y) & = & \varphi(y, \psi_2(y)), \quad y \in (-\varepsilon_0, \varepsilon_0) \\ Supp(E\varphi) & \subset & (-2/\varepsilon_0, 2/\varepsilon_0). \end{array} \right. \tag{22}$$

Proof: Given $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$, we first define $\tilde{\varphi} \in \mathcal{C}^{0,\alpha}(I\!R)$ by

$$\tilde{\varphi}(y) \quad = \quad \left\{ \begin{array}{ll} \varphi(y, \psi_2(y)), & \text{if } y \in [-\varepsilon_0, \varepsilon_0] \\ \varphi(\varepsilon_0, \psi_2(\varepsilon_0)), & \text{if } y > \varepsilon_0 \\ \varphi(-\varepsilon_0, \psi_2(-\varepsilon_0)), & \text{if } y < -\varepsilon_0. \end{array} \right.$$

It is easy to check that $||\tilde{\varphi}||_{0,\alpha} \leq ||\varphi||_{0,\alpha}$: For instance, when $|y| \leq \varepsilon_0$, $\hat{y} > \varepsilon_0$ and $|y - \hat{y}| < 1$, we can estimate

$$\begin{array}{lll} \dfrac{|\tilde{\varphi}(y) - \tilde{\varphi}(\hat{y})|}{|y-\hat{y}|^\alpha} & = & \dfrac{|\varphi(y, \psi_2(y)) - \varphi(\varepsilon_0, \psi_2(\varepsilon_0))|}{|y-\hat{y}|^\alpha} \\ & \leq & \dfrac{|\varphi(y, \psi_2(y)) - \varphi(\varepsilon_0, \psi_2(\varepsilon_0))|}{|y-\varepsilon_0|^\alpha} \ \leq \ ||\varphi||_{0,\alpha}, \end{array}$$

and similarly for the other choices of $y, \hat{y}$.

Next, let ρ denote a $C^1(I\!R)$ function with values in $[0, 1]$, with compact support in $(-\frac{2}{\varepsilon_0}, \frac{2}{\varepsilon_0})$, and such that

$$\left\{ \begin{array}{lll} \rho(y) & = & 1 \quad \text{if } |y| \leq \varepsilon_0 \\ ||\rho'||_\infty & \leq & \varepsilon_0. \end{array} \right.$$

We define $E\varphi(y) = \rho(y)\tilde{\varphi}(y)$, which satisfies $||E\varphi||_\infty \leq ||\tilde{\varphi}||_\infty \leq ||\varphi||_\infty$ and

$$\begin{array}{lll} \displaystyle\sup_{|y-\hat{y}|\leq 1} \frac{|E\varphi(y) - E\varphi(\hat{y})|}{|y-\hat{y}|^\alpha} & \leq & \displaystyle\sup_{|y-\hat{y}|\leq 1} \left(||\rho||_\infty \frac{|\tilde{\varphi}(y) - \tilde{\varphi}(\hat{y})|}{|y-\hat{y}|^\alpha} + ||\tilde{\varphi}||_\infty\, ||\rho'||_\infty |y-\hat{y}|^{1-\alpha} \right) \\ & \leq & (1+\varepsilon_0)||\varphi||_{0,\alpha}, \end{array}$$

and the lemma follows. ■

We let $\alpha < \alpha_0$ and fix $0 < \varepsilon < \varepsilon_0$. We introduce two auxiliairy functions $\psi_{1,\varepsilon}, \psi_{2,\varepsilon}$, defined on $I\!R$, which satisfy (see Figure 3) :

$$\psi_{j,\varepsilon} \ \equiv \ \psi_j, \qquad j = 1, 2, \ \ |x| \leq \varepsilon, \tag{23}$$

$$||\psi_{j,\varepsilon}||_{\mathcal{C}^{1,\alpha}} \ \leq \ 2||\psi_j||_{\mathcal{C}^{1,\alpha_0}} \varepsilon^\nu, \tag{24}$$

where $\nu = \alpha_0 - \alpha > 0$. The existence of such functions follows from the $\mathcal{C}^{1,\alpha_0}$ regularity of ψ_1 and ψ_2, and from the fact that

$$\psi_j(0) \;=\; \psi_j'(0) \;=\; 0.$$

We simply take

$$\psi_{j,\varepsilon}(x) \;=\; \begin{cases} \psi_j(x), & |x| < \varepsilon, \\ 2\psi_j(\pm\varepsilon) - \psi_j(\pm 2\varepsilon - x), & \varepsilon \le \pm x \le 2\varepsilon \\ 2\psi_j(\pm\varepsilon), & \pm x > 2\varepsilon. \end{cases}$$

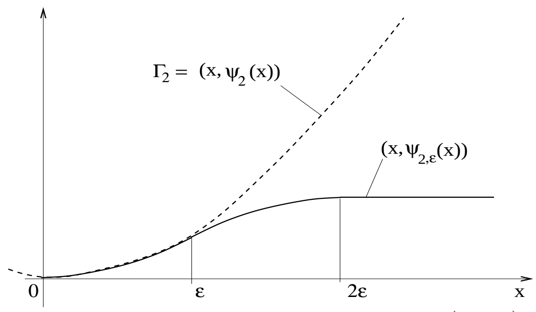

Figure 3: A possible contruction of $\psi_{2,\varepsilon}$: the part between $(\varepsilon, 2\varepsilon)$ is obtained by rotating the part between $(0,\varepsilon)$ around the point $(\varepsilon, \psi_2(\varepsilon))$.

Let $0 < \alpha < \alpha_0 \le 1$. Throughout the paper, we set for $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$

$$\phi(y) \;=\; E\varphi(y)\sqrt{1 + [\psi_{2,\varepsilon}'(y)]^2}.$$

It is easy to check that this function has regularity $\mathcal{C}^{0,\alpha}(I\!R)$ and that

$$||\phi||_{0,\alpha} \;\le\; (1 + C_\varepsilon)(1 + \varepsilon_0)||\varphi||_{\mathcal{C}^{0,\alpha}(\Gamma_2)}, \tag{25}$$

with $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

Let $\delta \ge 0$ and $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$. For $X \in \Gamma_1$ with first coordinate x, we set $X_\varepsilon = (x, \psi_{1,\varepsilon}(x))$. We also set $\Gamma_{2,\varepsilon} = \{Y = (y, \psi_{2,\varepsilon}(y)), y \in I\!R\}$. We then define

$$\begin{aligned} K_2^{\varepsilon,\delta}\varphi(X) \;=\; & \frac{-1}{2\pi}\int_{\Gamma_2} \frac{\nu(X)\cdot(X - Y - \delta e_2)}{|X - Y - \delta e_2|^2}\varphi(Y)\, d\sigma_Y \\ & + \frac{1}{2\pi}\int_{\Gamma_{2,\varepsilon}} \frac{\nu(X_\varepsilon)\cdot(X_\varepsilon - Y - \delta e_2)}{|X_\varepsilon - Y - \delta e_2|^2}\, E\varphi(Y)\, d\sigma_Y. \end{aligned} \tag{26}$$

More explicitely, the second term in the above expression has the form

$$\frac{1}{2\pi\sqrt{1 + [\psi_{1,\varepsilon}'(x)]^2}} \int_{I\!R} \frac{\begin{pmatrix} -\psi_{1,\varepsilon}'(x) \\ 1 \end{pmatrix}\cdot\begin{pmatrix} x - y \\ \psi_{1,\varepsilon}(x) - \psi_{2,\varepsilon}(y) - \delta \end{pmatrix}}{(x-y)^2 + (\delta + \psi_{2,\varepsilon}(y) - \psi_{1,\varepsilon}(x))^2}\, \phi(y) dy.$$

We remark that the two integrands in the definition of $K_2^{\varepsilon,\delta}$ coincide when $|y| \leq \varepsilon$ and $|x| \leq \varepsilon$, as $Y = (y, \psi_2(y)) = (y, \psi_{2,\varepsilon}(y))$ and $X = (x, \psi_1(x)) = (x, \psi_{1,\varepsilon}(x))$ in this case. We further define for $|X| \leq R_0$, and $\delta > 0$ or $X \neq 0$

$$\begin{aligned} J_2^{\varepsilon,\delta}\varphi(X) &= \int_{\mathbb{R}} \frac{(\delta + \psi_{2,\varepsilon}(y) - \psi_{1,\varepsilon}(x)) - \psi'_{1,\varepsilon}(x)(y-x)}{(x-y)^2 + (\delta + \psi_{2,\varepsilon}(y) - \psi_{1,\varepsilon}(x))^2}\,\phi(y)dy \\ &- \int_{\mathbb{R}} \frac{(\delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x)) - \psi'_{1,\varepsilon}(x)(y-x)}{(x-y)^2 + (\delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x))^2}\,\phi(y)dy, \end{aligned} \tag{27}$$

and for $\delta = 0$ and $X = 0$

$$J_2^{\varepsilon,0}\varphi(0) = \int_{\mathbb{R}} \frac{\psi_{2,\varepsilon}(y)}{y^2 + \psi_{2,\varepsilon}(y)^2}\phi(y)dy.$$

Note that the integral in the expression above is well-defined as $\psi_{2,\varepsilon}(y) = O(|y|^{1+\alpha})$ when $y \to 0$. Finally, for $|X| \leq R_0$, and $\delta > 0$ or $X \neq 0$, we define

$$I_2^{\varepsilon,\delta}\varphi(X) = \int_{\mathbb{R}} \frac{(\delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x)) - \psi'_{1,\varepsilon}(x)(y-x)}{(x-y)^2 + (\delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x))^2}\,\phi(y)dy, \tag{28}$$

and

$$I_2^{\varepsilon,0}\varphi(0) = \pi\varphi(0). \tag{29}$$

The expression $\frac{1}{2\pi\sqrt{1+(\psi'_{1,\varepsilon})^2}} I_2^{\varepsilon,\delta}$ represents the form one would (locally) have obtained for L_2^δ, if Γ_2 were a flat boundary at distance $\delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x)$ from Γ_1, see Figure 2.

Using the above definitions and recalling the definition (20) of χ, we may now decompose the off-diagonal operator $L_2^\delta, \delta > 0$, as follows

$$L_2^\delta\varphi(X) = \chi(X)L_2^\delta\varphi(X) + (1-\chi(X))L_2^\delta(X) \tag{30}$$

$$= \chi(X)\left(K_2^{\varepsilon,\delta} + \frac{1}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}(x)]^2}}(J_2^{\varepsilon,\delta} + I_2^{\varepsilon,\delta})\right)\varphi(X) \tag{31}$$

$$+(1-\chi(X))L_2^\delta\varphi(X). \tag{32}$$

In a similar manner, we define operators $K_1^{\varepsilon,\delta}, J_1^{\varepsilon,\delta}, I_1^{\varepsilon,\delta}$ from $\mathcal{C}^{0,\alpha}(\Gamma_1)$ into $\mathcal{C}^{0,\alpha}(\Gamma_2)$, that help decompose the operator L_1^δ

$$\begin{aligned} L_1^\delta &= \chi(X)\left(K_1^{\varepsilon,\delta} + \frac{1}{2\pi\sqrt{1+[\psi'_{2,\varepsilon}(x)]^2}}(J_1^{\varepsilon,\delta} + I_1^{\varepsilon,\delta})\right)\varphi(X) \\ &+(1-\chi(X))L_1^\delta\varphi(X), \end{aligned}$$

for $0 \le \varepsilon \le \varepsilon_0$.

The integral equation system (17) may now be written

$$T^\delta \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \;=\; \Lambda_{\varepsilon,\delta} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \;+\; C_{\varepsilon,\delta} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{33}$$

with

$$\Lambda_{\varepsilon,\delta} \;=\; \begin{pmatrix} \lambda I & \frac{\chi}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}(x)]^2}}(J_2^{\varepsilon,\delta}+I_2^{\varepsilon,\delta}) \\ \frac{\chi}{2\pi\sqrt{1+[\psi'_{2,\varepsilon}(x)]^2}}(J_1^{\varepsilon,\delta}+I_1^{\varepsilon,\delta}) & \lambda I \end{pmatrix} \tag{34}$$

and

$$C_{\varepsilon,\delta} \;=\; \begin{pmatrix} -K_1^* & \chi K_2^{\varepsilon,\delta} \\ \chi K_1^{\varepsilon,\delta} & -K_2^* \end{pmatrix} + (1-\chi)\begin{pmatrix} 0 & L_2^\delta \\ L_1^\delta & 0 \end{pmatrix}. \tag{35}$$

For $\delta = 0$, and $X \in \Gamma_1$, with $|X| > \varepsilon_0$, the definition (18) is used to define an auxiliary operator $\tilde{L}_2^0\varphi(X)$, *i.e.*,

$$\tilde{L}_2^0\varphi(X) \;=\; -\frac{\partial}{\partial\nu}S_2\varphi(X) \qquad X \in \Gamma_1,\ |X| > \varepsilon_0. \tag{36}$$

Since the single layer potential $S_2\varphi$ is infinitely regular away from the curve Γ_2, we have that $\tilde{L}_2^0\varphi = \lim_{\delta\to 0} L_2^\delta\varphi$ in $C^{0,\alpha}(\Gamma_1 \cap \{|X| > \varepsilon_0\})$, and as a consequence it follows immediately that

$$(1-\chi)L_2^\delta \;\to\; (1-\chi)\tilde{L}_2^0, \qquad \text{as } \delta \to 0,$$

in operator norm, from $C^{0,\alpha}(\Gamma_2)$ to $C^{0,\alpha}(\Gamma_1)$. We also note that the operators $(1-\chi)L_2^\delta$ and $(1-\chi)\tilde{L}_2^0$ are compact. We now define a global operator by

$$\begin{aligned} L_2^0\varphi(X) \;=\; & \chi(X)\left(K_2^{\varepsilon,0} + \frac{1}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}(x)]^2}}(J_2^{\varepsilon,0}+I_2^{\varepsilon,0})\right)\varphi(X) \\ & +(1-\chi(X))\tilde{L}_2^0\varphi(X). \end{aligned}$$

The operator L_2^0 is independent of ε and ε_0, since it is, as we shall show (in Lemma 8), the pointwise limit of the ε, ε_0-independent operator L_2^δ, as $\delta \to 0$. For that same reason $L_2^0\varphi(X)$ is also given by the formula (36) for $X \neq 0$. However, as we used the former to define the latter, different notation seems appropriate. A similar approach yields an ε, ε_0- independent operator L_1^0. We use the operators L_i^0, $i = 1,2$ to define the system

$$T^0 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \;=\; \begin{pmatrix} \lambda I - K_1^* & L_2^0 \\ L_1^0 & \lambda I - K_2^* \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{37}$$

As we shall show (in Lemma 8) this is indeed the limiting system corresponding to (17) as $\delta \to 0$. Due to the definition of T^0 it is easy to see that this operator may be decomposed as

$$T^0 \quad = \quad \Lambda_{\varepsilon,0} + C_{\varepsilon,0},$$

with

$$\Lambda_{\varepsilon,0} \quad = \quad \begin{pmatrix} \lambda I & \frac{\chi}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}(x)]^2}}(J_2^{\varepsilon,0} + I_2^{\varepsilon,0}) \\ \frac{\chi}{2\pi\sqrt{1+[\psi'_{2,\varepsilon}(x)]^2}}(J_1^{\varepsilon,0} + I_1^{\varepsilon,0}) & \lambda I \end{pmatrix} \tag{38}$$

and

$$C_{\varepsilon,0} \quad = \quad \begin{pmatrix} -K_1^* & \chi K_2^{\varepsilon,0} \\ \chi K_1^{\varepsilon,0} & -K_2^* \end{pmatrix} + (1-\chi)\begin{pmatrix} 0 & \tilde{L}_2^0 \\ \tilde{L}_1^0 & 0 \end{pmatrix}. \tag{39}$$

3 Main results

Our main goal is to show that the system of integral equations (17) is invertible, uniformly with respect to δ. As already discussed, all involved operators do not converge in norm as $\delta \to 0$, and the limiting system (37) is not of the form λ times the identity plus a compact perturbation. The single layer potentials K_i^* are compact operators on $\mathcal{C}^{0,\alpha}(\Gamma_i)$, as the curves Γ_i have regularity $\mathcal{C}^{1,\alpha}$ [15]. However, the off–diagonal terms L_i^0 are not quite as nice, even though their singular parts concentrate near only one point.

Lemma 4 *The operators L_2^0 and L_1^0 are not compact on $\mathcal{C}^{0,\alpha}$ for any $0 < \alpha < \alpha_0$.*

This result immediately implies that the compact operators (L_1^δ, L_2^δ) do not converge in norm to (L_1^0, L_2^0), and this eliminates a simple proof of uniform invertibility of (17). In order to overcome this difficulty, and still prove the uniform boundedness and convergence of the solutions to (17), we base our analysis on the decomposition (33), and use some fairly basic results from the theory of collectively compact operators, [6].

Definition 1 *Suppose X and Y are two Banach spaces. A family of compact linear operators $B^\delta : X \to Y$, $0 < \delta < \delta_0$ is called collectively compact if and only if the set $\{B^\delta\varphi, \quad ||\varphi||_X = 1,\ 0 < \delta < \delta_0\}$ is precompact in Y.*

The next three lemmas describe some important properties of the operators in the decomposition (33) of our system of integral equations. We only give the statements for the operators indexed by 2 (*i.e.*, those defined on $C^{0,\alpha}(\Gamma_2)$) but similar statements hold for the operators indexed by 1.

Lemma 5 *Let ε be fixed with $0 < \varepsilon < \varepsilon_0$. The operators $\chi K_2^{\varepsilon,\delta} : \mathcal{C}^{0,\alpha}(\Gamma_2) \longrightarrow \mathcal{C}^{0,\alpha}(\Gamma_1)$, $0 < \delta < \delta_0$ form a collectively compact family of operators.*

Lemma 6 *Given any $0 < \varepsilon < \varepsilon_0$ and any $0 \leq \delta < \delta_0$, the operator $\chi J_2^{\varepsilon,\delta}$ is a continuous linear operator from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1)$, $\alpha < \alpha_0$. Moreover, we have*

$$||\frac{\chi}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}]^2}} J_2^{\varepsilon,\delta}||_{\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_2),\mathcal{C}^{0,\alpha}(\Gamma_1))} \quad \leq \quad C(\varepsilon),$$

where $C(\varepsilon)$ converges to 0 uniformly in δ.

The operator $I_2^{\varepsilon,\delta}$ contains the most singular part of the off-diagonal term, however, it is possible to give a very precise estimate of the the size of this singular part in terms of $|\lambda|$.

Lemma 7 *Given any $0 < \varepsilon < \varepsilon_0$ and any $0 \leq \delta < \delta_0$, the operator $\chi I_2^{\varepsilon,\delta}$ is a continuous linear operator from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1)$, $\alpha < \alpha_0$. Furthermore we have the estimate*

$$||\frac{\chi}{2\pi\sqrt{1+[\psi'_{1,\varepsilon}]^2}} I_2^{\varepsilon,\delta}||_{\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_2),\mathcal{C}^{0,\alpha}(\Gamma_1))} \quad \leq \quad 1/2\,(1+C(\varepsilon))(1+\varepsilon_0), \tag{40}$$

where $C(\varepsilon) \to 0$, as $\varepsilon \to 0$, uniformly in δ.

The next statement concerns the pointwise convergence of the operators T^δ, as operators from $\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$.

Lemma 8 *Let $0 < \alpha < \alpha_0$, and fix $0 < \varepsilon < \varepsilon_0$. Then, as $\delta \to 0$, for all $(\varphi_1, \varphi_2) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$,*

$$K_1^{\varepsilon,\delta}\varphi_1, K_2^{\varepsilon,\delta}\varphi_2 \quad \longrightarrow \quad K_1^{\varepsilon,0}\varphi_1, K_2^{\varepsilon,0}\varphi_2, \quad \textit{in}\, \mathcal{C}^{0,\alpha}.$$

Additionnally, as $\delta \to 0$, for all $(\varphi_1, \varphi_2) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$,

$$\left\{ \begin{array}{lcl} \chi J_1^{\varepsilon,\delta}\varphi_1,\ \chi I_1^{\varepsilon,\delta}\varphi_1 & \longrightarrow & \chi J_1^{\varepsilon,0}\varphi_1,\ \chi I_1^{\varepsilon,0}\varphi_1 \\ \chi J_2^{\varepsilon,\delta}\varphi_2,\ \chi I_2^{\varepsilon,\delta}\varphi_2 & \longrightarrow & \chi J_1^{\varepsilon,0}\varphi_2,\ \chi I_2^{\varepsilon,0}\varphi_2, \end{array} \right. \qquad \textit{in}\ \mathcal{C}^{0,\alpha'},\ \alpha' < \alpha.$$

Consequently, since we already know that $(1-\chi)L_i^\delta\varphi_i \to (1-\chi)\tilde{L}_i^0\varphi_i$, in $\mathcal{C}^{0,\alpha}$, $i = 1, 2$, it follows that as $\delta \to 0$,

$$\begin{array}{rcll} \Lambda_{\varepsilon,\delta}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} & \to & \Lambda_{\varepsilon,0}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, & \textit{in}\ \mathcal{C}^{0,\alpha'}(\Gamma_1) \times \mathcal{C}^{0,\alpha'}(\Gamma_2),\ \alpha' < \alpha, \\ C_{\varepsilon,\delta}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} & \to & C_{\varepsilon,0}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, & \textit{in}\ \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2), \\ T^\delta\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} & \to & T^0\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} & \textit{in}\ \mathcal{C}^{0,\alpha'}(\Gamma_1) \times \mathcal{C}^{0,\alpha'}(\Gamma_2),\ \alpha' < \alpha. \end{array}$$

By the Uniform Boundedness Principle, the operators $C_{\varepsilon,\delta}$ are uniformly norm-bounded in $\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2))$.

The proofs of the lemmas stated above are given in the appendices A through C. We now state our main result.

Theorem 1 *Let $|\lambda| > 1/2$ and $\alpha < \alpha_0$. There exists $\delta_0 > 0$ such that the operators T^δ, $0 \leq \delta < \delta_0$, are invertible with inverses that are bounded independently of δ in $\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)), \alpha < \alpha_0$. Moreover, the operators $(T^\delta)^{-1}$ converge pointwise to $(T^0)^{-1}$ as $\delta \to 0$ in in $\mathcal{L}(\mathcal{C}^{0,\alpha'}(\Gamma_1) \times \mathcal{C}^{0,\alpha'}(\Gamma_2))$, for any $0 < \alpha' < \alpha$.*

Proof:
Step 1. Let $|\lambda| > 1/2$. Recall that we have tuned ε_0 so that $|\lambda| > (1+\varepsilon_0)/2$. Invoking Lemmas 6 and 7, we may fix $\varepsilon > 0$ sufficiently small that the off-diagonal terms of $\Lambda_{\varepsilon,\delta}$, being bounded in operator norm by $(1 + C(\varepsilon))(1 + \varepsilon_0)/2$, are strictly smaller than $|\lambda|$ uniformly for $0 \leq \delta \leq \delta_0$. Consequently, $\Lambda_{\varepsilon,\delta}$ is invertible in $\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2))$ and

$$\forall\, 0 \leq \delta \leq \delta_0, \quad ||\Lambda_{\varepsilon,\delta}^{-1}||_{\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_1)\times\mathcal{C}^{0,\alpha}(\Gamma_2))} \leq \frac{C}{|\lambda| - (1 + C(\varepsilon))(1 + \varepsilon_0)/2}, \tag{41}$$

with $C(\varepsilon) \to 0$, as $\varepsilon \to 0$, uniformly in δ. Further, it follows from Lemma 8, that for $(\varphi_1, \varphi_2) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$,

$$\Lambda_{\varepsilon,\delta}^{-1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \Lambda_{\varepsilon,0}^{-1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \quad \text{in } \mathcal{C}^{0,\alpha'},\ \alpha' < \alpha. \tag{42}$$

Step 2. As Γ_1 and Γ_2 are of regularity $\mathcal{C}^{1,\alpha+\nu}$, K_1^* and K_2^* are compact operators on $\mathcal{C}^{0,\alpha}(\Gamma_1)$ and $\mathcal{C}^{0,\alpha}(\Gamma_2)$ respectively (see for instance [15]). By Lemma 5 and Lemma 8 $C_{\varepsilon,0}$ is the strong limit of the collectively compact family of operators $C_{\varepsilon,\delta}$, and so it is also compact. In summary the operator $T^0 = \Lambda_{\varepsilon,0} + C_{\varepsilon,0}$ is a Fredholm operator: it is therefore invertible if proven injective.

Step 3. Let $(\varphi_1, \varphi_2) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$, such that

$$T^0 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0. \tag{43}$$

By Lemma 8, $L_2^\delta \varphi_2 \to L_2^0 \varphi_2$ as $\delta \to 0$, and so

$$\begin{aligned} \int_{\Gamma_1} L_2^0 \varphi_2 \, d\sigma &= \lim_{\delta\to 0} \int_{\Gamma_1} L_2^\delta \varphi_2 \, d\sigma \\ &= -\lim_{\delta\to 0} \int_{\Gamma_1} \partial_\nu S_2 \varphi_2^\delta (X - \delta e_2) d\sigma_X. \end{aligned}$$

Since $S_2\varphi_2^\delta(X-\delta e_2)$ is harmonic in D_1, the integrals on the last right–hand side vanish, and so do their limits. Invoking well known results in potential theory [16], we now get

$$\begin{aligned}\int_{\Gamma_1}\left[(\lambda I - K_1^*)\varphi_1 \; + L_2^0\varphi_2\right]\,d\sigma &= \int_{\Gamma_1}(\lambda I - K_1^*)\varphi_1\,d\sigma \\ &= (\lambda-1/2)\int_{\Gamma_1}\varphi_1\,d\sigma \quad .\end{aligned}$$

A similar relation for $(\lambda I - K_2^*)\varphi_2 + L_1^0\varphi_1$ holds on Γ_2. Thus, as a consequence of (43) and of the fact that $|\lambda|>1/2$,

$$\int_{\Gamma_1}\varphi_1\,d\sigma \;=\; \int_{\Gamma_2}\varphi_2\,d\sigma \;=\; 0 \quad . \tag{44}$$

Step 4. Consider the function w_0 defined on $I\!R^2\setminus(\Gamma_1\cup\Gamma_2)$ by

$$w_0 \;=\; S_1\varphi_1 + S_2\varphi_2. \tag{45}$$

We claim that $w_0\equiv 0$ in $I\!R^2$. Indeed, $S_1\varphi_1$ and $S_2\varphi_2$ are continuous functions on $I\!R^2$ and harmonic in $I\!R^2\setminus\Gamma_1$ and $I\!R^2\setminus\Gamma_2$ respectively. The regularity of Γ_1 and Γ_2 implies that $\nabla S_1\varphi_1$ and $\nabla S_2\varphi_2$ are bounded. Thus, w_0 is piecewise harmonic in $I\!R^2\setminus(\Gamma_1\cup\Gamma_2)$, with ∇w_0 piecewise continous and bounded. In particular, $w_0\in H^1_{loc}(I\!R^2)$. We note further that (43) expresses the continuity of $a_0\partial_n w_0$ across Γ_1 and Γ_2, except possibly at 0, and consequently w_0 is a local solution to

$$\operatorname{div}(a_0\nabla w_0) \;=\; 0 \quad \text{in } I\!R^2\setminus\{0\}. \tag{46}$$

As for the behavior of w_0 at infinity, a classical estimate of the Newtonian potential [16], under condition (44), yields

$$w_0(X) \;=\; O(|X|^{-1}) \;\;, \qquad \nabla w_0(X) \;=\; O(|X|^{-2}) \quad \text{for } |X|\to\infty \quad . \tag{47}$$

Let $0<\rho<R$ and let B_ρ and B_R denote the balls of radii ρ and R, centered at 0. We compute

$$\int_{B_R} a_0|\nabla w_0|^2 \;=\; \int_{B_R\setminus B_\rho} a_0|\nabla w_0|^2 \;+\; \int_{B_\rho} a_0|\nabla w_0|^2 \tag{48}$$

As w_0 is a_0–harmonic away from 0, the first integral reduces to

$$\begin{aligned}&\left|\int_{\partial B_R} a_0 w_0\partial_r w_0\,d\sigma \;-\; \int_{\partial B_\rho} a_0 w_0\partial_r w_0\,d\sigma\right| \\ &\leq\; C\int_{\partial B_R} R^{-3}\,d\sigma \;+\; ||w_0||_{L^\infty(\partial B_\rho)}\,||a_0\nabla w_0||_{L^\infty(\partial B_\rho)}|\partial B_\rho| \\ &\leq\; C\,R^{-2} + C\,\rho \;\;,\end{aligned}$$

where C is independent of R and ρ. We estimate the second integral by

$$\left| \int_{B_\rho} a_0 \nabla w_0 \cdot \nabla w_0 \right| \;\leq\; ||a_0||_{L^\infty(\mathbb{R}^2)} \, ||\nabla w_0||^2_{L^\infty(B_\rho)} |B_\rho| \;\leq\; C\rho^2.$$

Letting $R \to \infty$ and $\rho \to 0$ in (48), we conclude that $\int_{\mathbb{R}^2} a_0 |\nabla w_0|^2 \;=\; 0$, and in view of (47) that $w_0 \equiv 0$.

We now use the jump conditions for the single layer potential to obtain

$$\left\{ \begin{array}{lcll} \varphi_1(X) & = & \partial_\nu w_0^+ - \partial_\nu w_0^- & = 0, \quad X \in \Gamma_1 \setminus \{0\} \\ \varphi_2(X) & = & \partial_\nu w_0^+ - \partial_\nu w_0^- & = 0, \quad X \in \Gamma_2 \setminus \{0\}, \end{array} \right.$$

which together with the continuity of the φ_i's at 0 yields that $\varphi_1 = \varphi_2 \equiv 0$, *i.e.*, T^0 is injective.

Step 5. At this point we have verified that $\Lambda^{\varepsilon,0}$ and $\Lambda^{\varepsilon,\delta}$ are invertible for ε sufficiently small, the latter with inverses whose operator norms are bounded independently of δ. We next claim that

(i) The operators $C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta}$ are collectively compact.

(ii) $C_{\varepsilon,0}\Lambda^{-1}_{\varepsilon,0}$ is compact.

(iii) $C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta} \;\to\; C_{\varepsilon,0}\Lambda^{-1}_{\varepsilon,0}$ pointwise in $\mathcal{L}(\mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2))$ as $\delta \to 0$.

Under these conditions, Theorem 1.6 in [6] states that the operators $(I + C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta})^{-1}$ exist, for δ sufficiently small, and are bounded uniformly in δ if and only if $I + C_{\varepsilon,0}\Lambda^{-1}_{\varepsilon,0}$ is invertible. Moreover in that case

$$(I + C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta})^{-1} \quad\to\quad (I + C_{\varepsilon,0}\Lambda^{-1}_{\varepsilon,0})^{-1} \quad \text{pointwise} \; . \tag{49}$$

Since $T^\delta \;=\; (I + C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta})\Lambda_{\varepsilon,\delta}$, and since we already know that T^0 and $\Lambda^{\varepsilon,0}$ are invertible, the validity of the claims (i)–(iii) will thus let us conclude that $(I + C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta})^{-1}$ are uniformly norm bounded and that (49) holds. In combination with (41), (42) it follows that

$$(T^\delta)^{-1} \;=\; \Lambda^{-1}_{\varepsilon,\delta}(I + C_{\varepsilon,\delta}\Lambda^{-1}_{\varepsilon,\delta})^{-1}$$

are uniformly norm bounded, and satisfy

$$(T^\delta)^{-1} \;\to\; (T^0)^{-1} \quad \text{pointwise as } \delta \to 0 \; .$$

It therefore only remains to verify the claims (i)–(iii) in order to complete the proof of Theorem 1. As already noticed in Step 2, it follows directly from Lemma 5 and

Lemma 8 that the operators $C_{\varepsilon,\delta}$ form a collectively compact family and that the limit $C_{\varepsilon,0}$ is compact. The uniform bounds (41) now imply that the operators $C_{\varepsilon,\delta}\Lambda_{\varepsilon,\delta}^{-1}$ also form a collectively compact family. This verifies the claims (i) and (ii).

Since the operators $C_{\varepsilon,\delta}$ are collectively compact in $\mathcal{C}^{0,\alpha}(\Gamma_2)\times\mathcal{C}^{0,\alpha}(\Gamma_1)$, and since $(\Lambda_{\varepsilon,\delta}^{-1}-\Lambda_{\varepsilon,0}^{-1})\varphi$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Gamma_2)\times\mathcal{C}^{0,\alpha}(\Gamma_1)$, a subsequence of $C_{\varepsilon,\delta}(\Lambda_{\varepsilon,\delta}^{-1}-\Lambda_{\varepsilon,0}^{-1})\varphi$ converges to some function $w\in\mathcal{C}^{0,\alpha}(\Gamma_2)\times\mathcal{C}^{0,\alpha}(\Gamma_1)$. However, in view of (42), and of the fact that the operators $C_{\varepsilon,\delta}$ are norm-bounded in $\mathcal{L}(\mathcal{C}^{0,\alpha'}(\Gamma_1)\times\mathcal{C}^{0,\alpha'}(\Gamma_2))$, this subsequence must converge to 0 in $\mathcal{C}^{0,\alpha'}(\Gamma_1)\times\mathcal{C}^{0,\alpha'}(\Gamma_2)$, $0<\alpha'<\alpha$. Uniqueness of the limit implies that $w\equiv 0$, i.e., that $C_{\varepsilon,\delta}(\Lambda_{\varepsilon,\delta}^{-1}-\Lambda_{\varepsilon,0}^{-1})\varphi\to 0$ in $\mathcal{C}^{0,\alpha}(\Gamma_2)\times\mathcal{C}^{0,\alpha}(\Gamma_1)$. Since this is true for any subsequence, the whole sequence $C_{\varepsilon,\delta}(\Lambda_{\varepsilon,\delta}^{-1}-\Lambda_{\varepsilon,0}^{-1})\varphi$ converges to 0 in $\mathcal{C}^{0,\alpha}(\Gamma_2)\times\mathcal{C}^{0,\alpha}(\Gamma_1)$.

We then write

$$(C_{\varepsilon,\delta}\Lambda_{\varepsilon,\delta}^{-1}-C_{\varepsilon,0}\Lambda_{\varepsilon,0}^{-1})\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \;=\; \Big[C_{\varepsilon,\delta}(\Lambda_{\varepsilon,\delta}^{-1}-\Lambda_{\varepsilon,0}^{-1}) \;+\; (C_{\varepsilon,\delta}-C_{\varepsilon,0})\Lambda_{\varepsilon,0}^{-1}\Big]\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

to conclude that $C_{\varepsilon,\delta}\Lambda_{\varepsilon,\delta}^{-1}$ converges pointwise to $C_{\varepsilon,0}\Lambda_{\varepsilon,0}^{-1}$ in $\mathcal{C}^{0,\alpha}(\Gamma_2)\times\mathcal{C}^{0,\alpha}(\Gamma_1)$ as $\delta\to 0$, and therefore that (iii) holds. ■

Recall that the solution to the conduction problem where the inclusions are δ apart has the representation (14), in terms of the solutions $(\varphi_1^\delta,\varphi_2^\delta)$ to (15) and the harmonic function H_δ from (9). A similar relationship holds between the solution u_0 to the conduction problem with touching inclusions and the solutions $(\varphi_1^0,\varphi_2^0)$ to

$$T^0\begin{pmatrix}\varphi_1^0\\ \varphi_2^0\end{pmatrix} \;=\; \begin{pmatrix}\partial_\nu H_{0/\Gamma_1}\\ \partial_\nu H_{0/\Gamma_2}\end{pmatrix}\quad, \tag{50}$$

where H_0 is the harmonic function from (9). This is the assertion of the following theorem.

Proposition 1 *The solution u_0, to (5), may be written*

$$u_0(X) \;=\; S_1\varphi_1^0(X) \;+\; S_2\varphi_2^0(X) \;+\; H_0(X) \quad X\in\Omega\quad, \tag{51}$$

where H_0 is harmonic inside Ω, and defined by (9), and where the pair $(\varphi_1^0,\varphi_2^0)\in\mathcal{C}^{0,\alpha}(\Gamma_1)\times\mathcal{C}^{0,\alpha}(\Gamma_2)$ is the unique solution to (50).

Proof:
Since H_0 is harmonic inside Ω, and since Γ_1 and Γ_2 are $\mathcal{C}^{1+\alpha_0}$, the right–hand side of (50) lies in $\mathcal{C}^{0,\alpha}(\Gamma_1)\times\mathcal{C}^{0,\alpha}(\Gamma_2)$. By Theorem 1, the integral equation (50) therefore has

a unique solution $(\varphi_1^0, \varphi_2^0) \in \mathcal{C}^{0,\alpha}(\Gamma_1) \times \mathcal{C}^{0,\alpha}(\Gamma_2)$, for any $0 < \alpha < \alpha_0$. By Lemma 1, $\partial_\nu H_{\delta/\Gamma_i} \to \partial_\nu H_{0/\Gamma_i}$ in $\mathcal{C}^{0,\alpha}(\Gamma_i)$, and so we infer from Theorem 1 that

$$\begin{aligned}
\begin{pmatrix} \varphi_1^\delta \\ \varphi_2^\delta \end{pmatrix} - \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix} &= (T^\delta)^{-1} \begin{pmatrix} \partial_\nu H_{\delta/\Gamma_1} \\ \partial_\nu H_{\delta/\Gamma_2} \end{pmatrix} - (T^0)^{-1} \begin{pmatrix} \partial_\nu H_{0/\Gamma_1} \\ \partial_\nu H_{0/\Gamma_2} \end{pmatrix} \\
&= (T^\delta)^{-1} \left[\begin{pmatrix} \partial_\nu H_{\delta/\Gamma_1} \\ \partial_\nu H_{\delta/\Gamma_2} \end{pmatrix} - \begin{pmatrix} \partial_\nu H_{0/\Gamma_1} \\ \partial_\nu H_{0/\Gamma_2} \end{pmatrix} \right] \\
&\quad + \left[(T^\delta)^{-1} - (T^0)^{-1} \right] \begin{pmatrix} \partial_\nu H_{0/\Gamma_1} \\ \partial_\nu H_{0/\Gamma_2} \end{pmatrix} \\
&\to 0 \quad \text{in } \mathcal{C}^{0,\alpha'}(\Gamma_1) \times \mathcal{C}^{0,\alpha'}(\Gamma_2), \quad 0 < \alpha' < \alpha.
\end{aligned}$$

This convergence of φ_i^δ immediately implies that

$$S_1\varphi_1^\delta(X + \frac{\delta}{2}e_2) \to S_1\varphi_1^0(X), \quad \text{and} \quad S_2\varphi_2^\delta(X - \frac{\delta}{2}e_2) \to S_2\varphi_2^0(X), \tag{52}$$

uniformly on **compact** subdomains of $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, as $\delta \to 0$. Consider now the solution to the conduction problem (6), $u_\delta(X) = S_1\varphi_1^\delta(X + \frac{\delta}{2}e_2) + S_2\varphi_2^\delta(X - \frac{\delta}{2}e_2) + H_\delta(X)$. From Lemma 1 we know that $H_\delta \to H_0$ uniformly on compact subdomains of Ω, and if we combine this with (52) we obtain

$$u_\delta(X) = S_1\varphi_1^\delta(X + \frac{\delta}{2}e_2) + S_2\varphi_2^\delta(X - \frac{\delta}{2}e_2) + H_\delta(X) \to S_1\varphi_1^0(X) + S_2\varphi_2^0(X) + H_0(X) ,$$

uniformly on **compact** subdomains of $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, as $\delta \to 0$. Since we also know that $u_\delta \to u_0$ in $H^1(\Omega)$, it follows from the uniqueness of the limit that $u_0 = S_1\varphi_1^0 + S_2\varphi_2^0 + H_0$ on compact subdomains on $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$. Both sides of this identity are continuous functions, and so we get $u_0(X) = S_1\varphi_1^0(X) + S_2\varphi_2^0(X) + H_0(X)$ for all $X \in \Omega$, just as desired. ∎

The representation formula (51) of the previous theorem guarantees that u_0 and its gradient are piecewise smooth functions in Ω_η, and uniformly bounded. This property is transmitted to the solutions u_δ, as expressed in the following Theorem, an entirely different proof of which was already given in [20].

Theorem 2 *Let $\eta > 0$ and $0 < \alpha < \alpha_0$. The solutions u_δ to (6) satisfy*

$$||u_\delta||_{C^{1,\alpha}(\Omega_\eta \setminus \overline{(D_1^\delta \cup D_2^\delta)})} + ||u_\delta||_{C^{1,\alpha}(\overline{D_1^\delta})} + ||u_\delta||_{C^{1,\alpha}(\overline{D_2^\delta})} \leq C||g||_{L^2(\partial\Omega)}.$$

The constant C depends on η, but is independent of δ and g.

Proof:
Recall that u_δ has the representation

$$u_\delta(X) = S_1\varphi_1^\delta(X + \frac{\delta}{2}e_2) + S_2\varphi_2^\delta(X - \frac{\delta}{2}e_2) + H_\delta(X),$$

where $(\varphi_1^\delta, \varphi_2^\delta)$ solves (15) in $\mathcal{C}^{0,\bar\alpha}(\Gamma_1) \times \mathcal{C}^{0,\bar\alpha}(\Gamma_2)$, for any $\alpha < \bar\alpha < \alpha_0$. Adapting the arguments developed for the case of $\mathcal{C}^2$ contours in[14], Theorems 2.13 and 2.16, one easily obtains that

$$||S_1\varphi_1^\delta(X+\frac{\delta}{2}e_2)||_{\mathcal{C}^{1,\alpha}(\overline{\Omega\setminus D_1^\delta})} + ||S_1\varphi_1^\delta(X+\frac{\delta}{2}e_2)||_{\mathcal{C}^{1,\alpha}(\overline{D_1^\delta})} \ \leq \ C\,||\varphi_1^\delta||_{\mathcal{C}^{0,\bar\alpha}(\Gamma_1)} \ ,$$

and similarly

$$||S_2\varphi_2^\delta(X-\frac{\delta}{2}e_2)||_{\mathcal{C}^{1,\alpha}(\overline{\Omega\setminus D_2^\delta})} + ||S_2\varphi_2^\delta(X-\frac{\delta}{2}e_2)||_{\mathcal{C}^{1,\alpha}(\overline{D_2^\delta})} \ \leq \ C\,||\varphi_2^\delta||_{\mathcal{C}^{0,\bar\alpha}(\Gamma_2)} \ .$$

Due to Theorem 1 and the fact that $(\varphi_1^\delta, \varphi_2^\delta)$ solves (15)

$$||\varphi_1^\delta||_{\mathcal{C}^{0,\bar\alpha}(\Gamma_1)} + ||\varphi_2^\delta||_{\mathcal{C}^{0,\bar\alpha}(\Gamma_2)} \leq C||H_\delta||_{C^{1,\bar\alpha}(\Omega_\eta)},$$

for η sufficiently small. At the same time, due to Lemma 1,

$$||H_\delta||_{C^{1,\bar\alpha}(\Omega_\eta)} \leq C||g||_{L^2(\partial\Omega)}.$$

A combination of these four estimates with the above representation formula for u_δ immediately gives the apriori estimates from the statement of this theorem. ∎

A Proof of Lemma 5

To simplify our exposition, we drop the index 2 on the operators $K_2^{\varepsilon,\delta}, J_2^{\varepsilon,\delta}, I_2^{\varepsilon,\delta}$. Considering the definitions of $\psi_{1,\varepsilon}, \psi_{2,\varepsilon}$, for $X \in \Gamma_1, |X| \leq \varepsilon/2$, the operator $K^{\varepsilon,\delta}$ is given by

$$\begin{aligned} K^{\varepsilon,\delta}\varphi(X) \ &= \ -\frac{1}{2\pi}\int_{\Gamma_2} \frac{\nu(X)\cdot(X-Y-\delta e)}{|X-Y-\delta e|^2}(1-\chi(y))\varphi(Y)\,d\sigma_Y \\ &- \frac{1}{2\pi}\int_{\Gamma_2\cap\{|y|>\varepsilon\}} \frac{\nu(X)\cdot(X-Y-\delta e)}{|X-Y-\delta e|^2}\chi(y)\varphi(Y)\,d\sigma_Y \\ &- \frac{1}{2\pi\sqrt{1+[\psi_1'(x)]^2}}\int_{I\!R\cap\{|y|>\varepsilon\}} \frac{(\delta+\psi_{2,\varepsilon}(y)-\psi_{1,\varepsilon}(x))-\psi_{1,\varepsilon}'(x)(y-x)}{(x-y)^2+(\delta+\psi_{2,\varepsilon}(y)-\psi_{1,\varepsilon}(x))^2}\ \phi(y)\,dy. \end{aligned}$$

Since for $|y| > \varepsilon$ and $|x| \leq \varepsilon/2$, $(x-y)^2 \geq \varepsilon^2/4$, one easily checks that the kernels in all the above integrals are bounded and have regularity $\mathcal{C}^{0,\alpha_0}$, so that $K^{\delta,\varepsilon}$ is compact and maps $\mathcal{C}^{0,\alpha}(\Gamma_2)$ into $\mathcal{C}^{0,\alpha}(\Gamma_1 \cap \{|X| < \varepsilon/2\})$, for any $0 < \alpha < \alpha_0$. An even more direct argument works for $|X| > \varepsilon/2$. We also note that the bounds on the kernels are uniform with respect to $0 \leq \delta \leq 1$. As a consequence, the operators $K^{\varepsilon,\delta}, 0 \leq \delta < 1$ form a family of collectively compact operators.

∎

B Proof of Lemma 6

Recall that we assumed ψ_1, ψ_2 have regularity $\mathcal{C}^{0,\alpha_0}$ for some $0 < \alpha_0 \leq 1$. Let $\alpha < \alpha_0$ with $\nu = \alpha_0 - \alpha > 0$. Our construction of the auxiliairy functions $\psi_{1,\varepsilon}$ and $\psi_{2,\varepsilon}$ implies that the following bound holds

$$||\psi_{1,\varepsilon}||_{1,\alpha}, ||\psi_{2,\varepsilon}||_{1,\alpha} \quad \leq \quad C\varepsilon^{\nu}.$$

In this section, we show that $J^{\varepsilon,\delta}$ maps $\mathcal{C}^{0,\alpha}(\Gamma_2)$ into $\mathcal{C}^{0,\alpha}(\Gamma_1)$ for any $0 < \alpha < \alpha_0$.

Given $s, x, \hat{x} \in I\!R$, we write henceforth

$$\begin{array}{rclcl} a & = & a(x) & = & \delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x) \\ \hat{a} & = & a(\hat{x}) & = & \delta + \psi_{2,\varepsilon}(\hat{x}) - \psi_{1,\varepsilon}(\hat{x}) \\ b & = & b(x,s) & = & \delta + \psi_{2,\varepsilon}(s+x) - \psi_{1,\varepsilon}(x) \\ \hat{b} & = & b(\hat{x},s) & = & \delta + \psi_{2,\varepsilon}(s+\hat{x}) - \psi_{1,\varepsilon}(\hat{x}). \end{array} \tag{53}$$

B.1 Preliminary estimates

We will repeatedly have to estimate differences such as

$$|b-a| \quad = \quad |\psi_{2,\varepsilon}(s+x) - \psi_{2,\varepsilon}(x)|.$$

The mean value theorem shows that for some θ between 0 and s

$$\begin{array}{rcl} |b-a| & = & |\psi'_{2,\varepsilon}(x+\theta)|\,|s| \\ & \leq & \left(|\psi'_{2,\varepsilon}(x)| \ + \ |\psi'_{2,\varepsilon}(x+\theta) - \psi'_{2,\varepsilon}(x)|\right)\ |s| \\ & \leq & \left(|\psi'_{2,\varepsilon}(x)| \ + \ ||\psi'_{2,\varepsilon}||_{0,\alpha}|\theta|^{\alpha}\right)\ |s| \\ & \leq & \left(|\psi'_{2,\varepsilon}(x)| \ + \ ||\psi'_{2,\varepsilon}||_{0,\alpha}|s|^{\alpha}\right)\ |s|. \end{array} \tag{54}$$

Alternatively, we may bound $|b-a|$ by

$$|b-a| \quad = \quad \left(|\psi'_{2,\varepsilon}(s+x)| \ + \ ||\psi'_{2,\varepsilon}||_{0,\alpha}|s|^{\alpha}\right)\ |s|. \tag{55}$$

Similar estimates can be derived for $|b-\hat{b}|$: setting $d = |x-\hat{x}|$, we have for some θ between x and $\hat{x}$

$$\begin{array}{rcl} |b-\hat{b}| & = & |\psi'_{2,\varepsilon}(s+\theta) - \psi'_{1,\varepsilon}(\theta)|\ d \\ & \leq & \left(|\psi'_{2,\varepsilon}(s+\hat{x})| + |\psi'_{1,\varepsilon}(\hat{x})| + |\psi'_{2,\varepsilon}(s+\theta) - \psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\theta) + \psi'_{1,\varepsilon}(\hat{x})|\right)\ d \\ & \leq & \left(|\psi'_{2,\varepsilon}(s+\hat{x})| + |\psi'_{1,\varepsilon}(\hat{x})| + d^{\alpha}(||\psi'_{2,\varepsilon}||_{0,\alpha} + ||\psi'_{1,\varepsilon}||_{0,\alpha})\right)\ d. \end{array} \tag{56}$$

Similar estimates hold for $|a-\hat{a}|$ and $|\hat{b}-\hat{a}|$.

Recall also that $\phi(y) = E\varphi(y)\sqrt{1+[\psi_2'(y)]^2}$ has support in $(-R_0, R_0)$. Thus, there exists $M > 0$ such that for any $X \in \Gamma_1$ with first coordinate x, the function $s \longrightarrow \phi(s+x)$ is supported in $(-M, M)$.

Our analysis relies on the following lower bound on $|\psi_{1,\varepsilon}|, |\psi_{2,\varepsilon}|$:

Proposition 2 *Suppose* $0 < \alpha \le \alpha_0$. *There exists a constant* $C > 0$, *independent of* ε, *such that for any* $x \in I\!R$,

$$|\psi_{i,\varepsilon}'(x)| \quad \le \quad C|\psi_{i,\varepsilon}(x)|^{\frac{\alpha}{1+\alpha}}, \quad i = 1, 2. \tag{57}$$

Proof: We only focus on ψ_2, but the same arguments apply to ψ_1. Recall that we assume Γ_2 is strictly convex, and that ψ_2 is $C^{1,\alpha}$ and positive, vanishing only at 0. The function ψ_2 is only defined in a neighborhood $(-\varepsilon_0, \varepsilon_0)$ around 0. We may nevertheless extend it on the whole of $I\!R$, as a $\mathcal{C}^{1,\alpha}$ function that only vanishes at 0 and such that $||\psi_2||_{1,\alpha,I\!R} \le 2||\psi_2||_{1,\alpha,(-\varepsilon_0,\varepsilon_0)}$. It follows that for any $x \in [-M, M]$ and for any $\theta \in I\!R$

$$\psi_2(x+\theta) \quad \le \quad \psi_2(x) + \psi_2'(x)\theta + O(|\theta|^{1+\alpha}),$$

so that for some constant $C > 0$, independent of θ.

$$\psi_2(x) + \psi_2'(x)\theta + C|\theta|^{1+\alpha} \ge 0.$$

As a function of θ, the left–hand side of the above expression is minimal when $\theta_0 = -(\frac{\psi_2'(x)}{C(1+\alpha)})^{1/\alpha}$ if $\psi_2'(x) \ge 0$, and when $\theta_0 = (\frac{|\psi_2'(x)|}{C(1+\alpha)})^{1/\alpha}$ if $\psi_2'(x) < 0$. In both cases, the positivity of ψ_2 yields (57) for the function ψ_2.

We note that (57) is therefore satisfied by $\psi_{2,\varepsilon}$ when $|x| < \varepsilon$. It is trivially satisfied when $|x| > 2\varepsilon$. Furthermore, when $\varepsilon \le x \le 2\varepsilon$ one has

$$\begin{aligned} |\psi_{2,\varepsilon}(x)| \quad &= \quad 2\psi_2(\varepsilon) - \psi_2(2\varepsilon - x) \ \ge \ \psi_2(2\varepsilon - x) \\ &\ge \quad C|\psi_2'(2\varepsilon - x)|^{\frac{1+\alpha}{\alpha}} \ = \ C|\psi_{2,\varepsilon}'(x)|^{\frac{1+\alpha}{\alpha}}. \end{aligned}$$

■

Proposition 3 *For any* $s, t \ge 0$ *and for any* $0 \le \mu \le 1$ *we have*

$$s^2 + t^2 \quad \ge \quad s^{1+\mu}t^{1-\mu}.$$

Proof: We may assume that $t > 0$ and $\mu < 1$. By homogeneity, it suffices to show that $g(s) := s^2 - s^{1+\mu} + 1 \ge 0$ for any $s \ge 0$. One easily checks that g' only vanishes at $s_0 = (\frac{1+\mu}{2})^{\frac{1}{1-\mu}}$ and that

$$g(s_0) \quad = \quad (\frac{1+\mu}{2})^{\frac{2}{1-\mu}} + 1 - (\frac{1+\mu}{2})^{\frac{1+\mu}{1-\mu}} \ > \ 0.$$

■

B.2 Uniform bound on $J^{\varepsilon,\delta}$, $\delta > 0$:

Let $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$, and $X \in \Gamma_1, |X| \leq R_0$, with first coordinate x. Let

$$j_{\varepsilon,\delta}(s,x) \;=\; \left(\frac{b(x,s) - s\psi'_{1,\varepsilon}(x)}{s^2 + b(x,s)^2} - \frac{a(x) - s\psi'_{1,\varepsilon}(x)}{s^2 + a(x)^2} \right),$$

so that after the change of variable $s = y - x$,

$$J^{\varepsilon,\delta}\varphi(X) \;=\; \int_{|s|<M} j_{\varepsilon,\delta}(s,x)\;\phi(s+x).$$

It follows that

$$\begin{aligned} |J^{\varepsilon,\delta}\varphi(X)| &\leq \int_{|s|<M} \left| \frac{b - s\psi'_{1,\varepsilon}(x)}{s^2 + b^2} - \frac{a - s\psi'_{1,\varepsilon}(x)}{s^2 + a^2} \right| \,||\phi||_{0,\alpha} \\ &\leq ||\phi||_{0,\alpha} \int_{|s|<M} \frac{|b-a|(s^2 + |ab| + |s\psi'_{1,\varepsilon}(x)|\,|a+b|)}{(s^2+b^2)(s^2+a^2)} \\ &\leq C\,||\phi||_{0,\alpha} \int_{|s|<M} \frac{|b-a|}{s^2+a^2} + \frac{|b-a|}{s^2+b^2}. \end{aligned}$$

Recalling (54)-(55), and using propositions 2 and 3, we see that

$$\begin{aligned} &\int_{|s|<M} \frac{|b-a|}{s^2+b^2} + \frac{|b-a|}{s^2+a^2} \\ &\leq \int_{|s|<M} \frac{|s|\left(|\psi'_{2,\varepsilon}(s+x)| + s^\alpha ||\psi'_{2,\varepsilon}||_{0,\alpha}\right)}{s^2+b^2} + \frac{|s|\left(|\psi'_{2,\varepsilon}(x)| + s^\alpha ||\psi'_{2,\varepsilon}||_{0,\alpha}\right)}{s^2+a^2} \\ &\leq C \int_{|s|<M} \frac{|s|\,|\psi_{2,\varepsilon}(s+x)|^{\frac{\alpha}{1+\alpha}}}{|s|^{1+\mu}\,|\psi_{2,\varepsilon}(s+x)+\delta|^{1-\mu}} + \frac{|s||\psi_{2,\varepsilon}(x)|^{\frac{\alpha}{1+\alpha}}}{|s|^{1+\mu}\,|\psi_{2,\varepsilon}(x)+\delta|^{1-\mu}} \\ &\quad + C\,||\psi'_{2,\varepsilon}||_{0,\alpha} \int_{|s|<M} |s|^{\alpha-1}. \end{aligned}$$

We choose μ such that $1/(1+\alpha) < \mu < 1$, and thus $\alpha/(1+\alpha) - (1-\mu) > 0$, to obtain

$$\begin{aligned} &\int_{|s|<M} \frac{|b-a|}{s^2+b^2} + \frac{|b-a|}{s^2+a^2} \\ &\leq C\,||\psi'_{2,\varepsilon}||_{0,\alpha}\,M^\alpha + C\,||\psi_{2,\varepsilon}||_\infty^{\frac{\alpha}{1+\alpha}-(1-\mu)} \int_{|s|<M} |s|^{-\mu} \\ &\leq C\left(||\psi'_{2,\varepsilon}||_{0,\alpha}\,M^\alpha + ||\psi_{2,\varepsilon}||_\infty^{\frac{\alpha}{1+\alpha}-(1-\mu)}\,M^{1-\mu} \right) \leq C(\varepsilon), \end{aligned} \tag{58}$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly with respect to δ, since $||\psi_{2,\varepsilon}||_{1,\alpha} = O(\varepsilon^\nu)$. Hence, recalling (25), we see that

$$|J^{\varepsilon,\delta}\varphi(X)| \;\leq\; C(\varepsilon)\;||\varphi||_{0,\alpha}, \tag{59}$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly with respect to δ (and X).

B.3 Hölder continuity of $J^{\varepsilon,\delta}$, $\delta > 0$

Let $X, \hat{X} \in \Gamma_1 \cap B(0, R_0)$, with respective abcissae $x, \hat{x}$ and set

$$d \;=\; |x - \hat{x}| \;\leq\; |X - \hat{X}|. \tag{60}$$

Using the notations of the previous section, we form

$$\begin{aligned}
&J^{\varepsilon,\delta}\varphi(X) - J^{\varepsilon,\delta}\varphi(\hat{X}) \\
&\quad = \int_{|s|<M} \left(\frac{b - s\psi'_{1,\varepsilon}(x)}{s^2 + b^2} - \frac{a - s\psi'_{1,\varepsilon}(x)}{s^2 + a^2} \right) [\phi(s + x) - \phi(s + \hat{x})] \\
&\qquad + \int_{|s|<M} (j_{\varepsilon,\delta}(s, x) - j_{\varepsilon,\delta}(s, \hat{x})) \, [\phi(s + \hat{x}) - \phi(\hat{x})] \\
&\qquad + \phi(\hat{x}) \int_{|s|<M} \left(\frac{b - s\psi'_{1,\varepsilon}(x)}{s^2 + b^2} - \frac{a - s\psi'_{1,\varepsilon}(x)}{s^2 + a^2} - \frac{\hat{b} - s\psi'_{1,\varepsilon}(\hat{x})}{s^2 + \hat{b}^2} + \frac{\hat{a} - s\psi'_{1,\varepsilon}(\hat{x})}{s^2 + \hat{a}^2} \right) \\
&\quad =: \; R_1 \;+\; R_2 \;+\; R_3.
\end{aligned} \tag{61}$$

B.3.1 Control of R_1

Using (54), it follows that

$$\begin{aligned}
|R_1| \;&=\; \left| \int_{|s|<M} \left(\frac{b - s\psi'_{1,\varepsilon}(x)}{s^2 + b^2} - \frac{a - s\psi'_{1,\varepsilon}(x)}{s^2 + a^2} \right) [\phi(s + x) - \phi(s + \hat{x})] \; ds \right| \\
&\leq\; \int_{|s|<M} \frac{|b - a| \; \left(s^2 + |ab| + |s(a + b)\psi'_{1,\varepsilon}(x)|\right)}{(s^2 + a^2)(s^2 + b^2)} \; ||\phi||_{0,\alpha} d^\alpha \\
&\leq\; C \; ||\phi||_{0,\alpha} \; d^\alpha \int_{|s|<M} \frac{|b - a|}{s^2 + b^2} \;+\; \frac{|b - a|}{s^2 + a^2},
\end{aligned}$$

and we conclude from (58) that

$$|R_1| \;\;\leq\;\; C(\varepsilon) ||\phi||_{0,\alpha} d^a, \tag{62}$$

with $C(\varepsilon) \to 0$ when $\varepsilon \to 0$, uniformly in δ (and $X, \hat{X}$).

B.3.2 Control of R_2

We rewrite R_2 as

$$\begin{aligned}
R_2 \;\;=\;\; &\int_{|s|<d} \left(\frac{b - s\psi'_{1,\varepsilon}(x)}{s^2 + b^2} - \frac{a - s\psi'_{1,\varepsilon}(x)}{s^2 + a^2} \right) \; [\phi(s + \hat{x}) - \phi(\hat{x})] \\
&- \int_{|s|<d} \left(\frac{\hat{b} - s\psi'_{1,\varepsilon}(\hat{x})}{s^2 + \hat{b}^2} - \frac{\hat{a} - s\psi'_{1,\varepsilon}(\hat{x})}{s^2 + \hat{a}^2} \right) \; [\phi(s + \hat{x}) - \phi(\hat{x})]
\end{aligned}$$

$$
\begin{aligned}
& + \int_{d<|s|<M} \left(\frac{b - s\psi'_{1,\varepsilon}(x)}{s^2+b^2} - \frac{\hat{b} - s\psi'_{1,\varepsilon}(\hat{x})}{s^2+\hat{b}^2} \right) [\phi(s+\hat{x}) - \phi(\hat{x})] \\
& - \int_{d<|s|<M} \left(\frac{a - s\psi'_{1,\varepsilon}(x)}{s^2+a^2} - \frac{\hat{a} - s\psi'_{1,\varepsilon}(\hat{x})}{s^2+\hat{a}^2} \right) [\phi(s+\hat{x}) - \phi(\hat{x})] \\
=: \; & S_1 + S_2 + S_3 + S_4. \qquad (63)
\end{aligned}
$$

The first term can be estimated by

$$
\begin{aligned}
|S_1| & \leq C \int_{|s|<d} \left(\frac{|b-a|}{s^2+b^2} + \frac{|b-a|}{s^2+a^2} \right) |s|^\alpha \, ||\phi||_{0,\alpha} \\
& \leq ||\phi||_{0,\alpha} \int_{|s|<d} \left(\frac{|s| \, ||\psi'_{2,\varepsilon}||_{0,\alpha}}{s^2+b^2} + \frac{|s| \, ||\psi'_{2,\varepsilon}||_{0,\alpha}}{s^2+a^2} \right) |s|^\alpha \\
& \leq C \, ||\phi||_{0,\alpha} \, ||\psi'_{2,\varepsilon}||_{0,\alpha} \int_{|s|<d} |s|^{\alpha-1} \\
& \leq C \, ||\phi||_{0,\alpha} \, ||\psi'_{2,\varepsilon}||_{0,\alpha} \, d^\alpha. \qquad (64)
\end{aligned}
$$

The same estimate holds for S_2.

Concerning S_3, we can rewrite the term in parentheses in the integrand as

$$
\frac{(b-\hat{b})(s^2 - b\hat{b})}{(s^2+b^2)(s^2+\hat{b}^2)} + \frac{s^3(\psi'_{1,\varepsilon}(\hat{x}) - \psi'_{1,\varepsilon}(x))}{(s^2+b^2)(s^2+\hat{b}^2)}
+ \frac{sb^2(\psi'_{1,\varepsilon}(\hat{x}) - \psi'_{1,\varepsilon}(x)) + s\psi'_{1,\varepsilon}(x)(b-\hat{b})(b+\hat{b})}{(s^2+b^2)(s^2+\hat{b}^2)}
$$

The estimate (56) then shows that

$$
\begin{aligned}
|S_3| \leq \; & \int_{d<|s|<M} \frac{|b-\hat{b}|}{s^2+\min(b,\hat{b})^2} |s|^\alpha \, ||\phi||_{0,\alpha} + \int_{d<|s|<M} 2 \, \frac{|s|^{1+\alpha} ||\psi'_{1,\varepsilon}||_{0,\alpha} d^\alpha}{s^2} \, ||\phi||_{0,\alpha} \\
& + \int_{d<|s|<M} \frac{||\psi'_{1,\varepsilon}||_\infty |b-\hat{b}|}{s^2+\min(b,\hat{b})^2} |s|^\alpha \, ||\phi||_{0,\alpha} \\
\leq \; & C \, ||\phi||_{0,\alpha} \int_{d<|s|<M} (1 + ||\psi'_{1,\varepsilon}||_\infty) \, \frac{\left(||\psi'_{2,\varepsilon}||_{0,\alpha} + ||\psi'_{1,\varepsilon}||_{0,\alpha}\right) d}{s^2} \, |s|^\alpha \\
& + C \, ||\phi||_{0,\alpha} \int_{d<|s|<M} ||\psi'_{1,\varepsilon}||_{0,\alpha} d^\alpha |s|^{\alpha-1} \\
\leq \; & C \, ||\phi||_{0,\alpha} (1 + ||\psi'_{1,\varepsilon}||_\infty) \left(||\psi'_{2,\varepsilon}||_{0,\alpha} + ||\psi'_{1,\varepsilon}||_{0,\alpha} \right) \left(d \int_{d<s<M} s^{\alpha-2} + M^\alpha d^\alpha \right) \\
\leq \; & C \, ||\phi||_{0,\alpha} (1 + ||\psi'_{1,\varepsilon}||_\infty) \left(||\psi'_{2,\varepsilon}||_{0,\alpha} + ||\psi'_{1,\varepsilon}||_{0,\alpha} \right) d^\alpha. \qquad (65)
\end{aligned}
$$

The same argument yields a similar estimate for S_4. In summary we obtain

$$
|R_2| \leq C \, \varepsilon^\nu \, ||\phi||_{0,\alpha} \, d^\alpha, \quad \nu = \alpha_0 - \alpha. \qquad (66)
$$

B.3.3 Control of R_3

The term R_3 is the most singular in (61). We rewrite it as $\phi(\hat{x})R_3'$ with

$$\begin{aligned} R_3' &= \int_{|s|<M} \left(\frac{b - s\psi_{2,\varepsilon}'(s+x)}{s^2+b^2} - \frac{\hat{b} - s\psi_{2,\varepsilon}'(s+\hat{x})}{s^2+\hat{b}^2} \right) \\ &\quad - \int_{|s|<M} \left(\frac{a}{s^2+a^2} - \frac{\hat{a}}{s^2+\hat{a}^2} \right) \\ &\quad + \int_{|s|<M} \left(\frac{s(\psi_{2,\varepsilon}'(s+x) - \psi_{1,\varepsilon}'(x))}{s^2+b^2} - \frac{s(\psi_{2,\varepsilon}'(s+\hat{x}) - \psi_{1,\varepsilon}'(\hat{x}))}{s^2+\hat{b}^2} \right) \\ &\quad + \int_{|s|<M} \left(\frac{s\psi_{1,\varepsilon}'(x)}{s^2+a^2} - \frac{s\psi_{1,\varepsilon}'(\hat{x})}{s^2+\hat{a}^2} \right) ds \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{67}$$

Noting that $\psi_{2,\varepsilon}'(s+x) = \partial_s b$ the first term can be integrated explicitely to obtain

$$T_1 = [\arctan(\frac{s}{b(s,x)}) - \arctan(\frac{s}{b(s,\hat{x})}]_{-M}^{M}$$

The mean value theorem shows that

$$\begin{aligned} |\arctan(\frac{M}{b(M,x)}) - \arctan(\frac{M}{b(M,\hat{x})})| &\le \frac{M\left(|\psi_{2,\varepsilon}'(M+\theta)| + |\psi_{1,\varepsilon}'(\theta)|\right)}{M^2 + b(M,\theta)^2} |x - \hat{x}| \\ &\le C\left(||\psi_{1,\varepsilon}'||_{0,\alpha} + ||\psi_{2,\varepsilon}'||_{0,\alpha}\right) d, \end{aligned} \tag{68}$$

and similarly with M replaced by $-M$. It follows that

$$|T_1| \le C(\varepsilon) d, \tag{69}$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in δ.

The term T_2 can be treated in the same fashion. Note also, that as $\frac{s}{s^2+a^2}$ is an odd function of s, $T_4 = 0$.

Finally, we decompose T_3 as follows:

$$\begin{aligned} T_3 &= \int_{d<|s|<M} \frac{s[\psi_{2,\varepsilon}'(s+x) - \psi_{1,\varepsilon}'(x) - \psi_{2,\varepsilon}'(s+\hat{x}) + \psi_{1,\varepsilon}'(\hat{x})]}{s^2+b^2} \\ &\quad + \int_{d<|s|<M} s[\psi_{2,\varepsilon}'(s+\hat{x}) - \psi_{1,\varepsilon}'(\hat{x})] \left(\frac{1}{s^2+b^2} - \frac{1}{s^2+\hat{b}^2} \right) \\ &\quad + \int_{|s|<d} \frac{s[\psi_{2,\varepsilon}'(s+x) - \psi_{1,\varepsilon}'(x)]}{s^2+b^2} - \int_{|s|<d} \frac{s[\psi_{2,\varepsilon}'(s+\hat{x}) - \psi_{1,\varepsilon}'(\hat{x})]}{s^2+\hat{b}^2} \\ &=: U_1 + U_2 + U_3 + U_4. \end{aligned} \tag{70}$$

Estimate for U_1:
The fact that $\psi'_{1,\varepsilon}$ and $\psi'_{2,\varepsilon}$ are $\mathcal{C}^{0,\beta}$ for any $\alpha < \beta \le \alpha_0$ gives

$$\begin{aligned} |U_1| &\le d^\beta (||\psi'_{1,\varepsilon}||_{0,\beta} + ||\psi'_{2,\varepsilon}||_{0,\beta}) \int_{d<|s|<M} \frac{|s|}{s^2 + (\psi_{1,\varepsilon}(x) - \delta)^2} \\ &\le d^\beta (||\psi'_{1,\varepsilon}||_{0,\beta} + ||\psi'_{2,\varepsilon}||_{0,\beta}) \, [\ln(s^2 + (\psi_{1,\varepsilon}(x) - \delta)^2)]_d^M \\ &\le C \, d^\beta (||\psi'_{1,\varepsilon}||_{0,\beta} + ||\psi'_{2,\varepsilon}||_{0,\beta}) \, |\ln(d^2)|, \end{aligned}$$

for d sufficiently small. Thus, we have

$$\begin{aligned} |U_1| &\le C \, (||\psi'_{1,\varepsilon}||_{0,\beta} + ||\psi'_{2,\varepsilon}||_{0,\beta}) d^\beta |\ln(d)| \\ &\le C \, \varepsilon^{\alpha_0 - \beta} \, d^\alpha, \end{aligned} \tag{71}$$

for any $\alpha < \beta \le \alpha_0$. This shows that

$$|U_1| \le C \, \varepsilon^\nu \, d^\alpha,$$

for any $0 < \nu < \alpha_0 - \alpha$, where C is independent of ε, d and δ.
Estimate for U_2:
To estimate U_2 we proceed as follows:

$$\begin{aligned} |U_2| &\le \int_{d<|s|<M} |s| \, \left|\psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\hat{x})\right| \frac{|b-\hat{b}| \, |b+\hat{b}|}{(s^2+b^2)(s^2+\hat{b}^2)} \\ &\le \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\hat{x})\right| \left(\frac{|b-\hat{b}|}{s^2+\hat{b}^2} + \frac{|b-\hat{b}|}{s^2+b^2} \right) \\ &\le \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\hat{x})\right| \frac{|b-\hat{b}|}{s^2+\hat{b}^2} \\ &\quad + \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+x) - \psi'_{1,\varepsilon}(x)\right| \frac{|b-\hat{b}|}{s^2+b^2} \\ &\quad + \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\hat{x}) - \psi'_{2,\varepsilon}(s+x) + \psi'_{1,\varepsilon}(x)\right| \frac{|b-\hat{b}|}{s^2+b^2}. \end{aligned}$$

Recalling (56) we obtain

$$\begin{aligned} |U_2| &\le \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\hat{x})\right| \frac{d\left(||\psi'_{1,\varepsilon}||_{0,\alpha} + ||\psi'_{2,\varepsilon}||_{0,\alpha}\right) \, d^\alpha}{s^2+\hat{b}^2} \\ &\quad + \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+\hat{x}) - \psi'_{1,\varepsilon}(\hat{x})\right| \frac{d\left(|\psi'_{2,\varepsilon}(s+\hat{x})| \; + \; |\psi'_{1,\varepsilon}(\hat{x})|\right)}{s^2+\hat{b}^2} \\ &\quad + \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+x) - \psi'_{1,\varepsilon}(x)\right| \frac{d\left(||\psi'_{1,\varepsilon}||_{0,\alpha} + ||\psi'_{2,\varepsilon}||_{0,\alpha}\right) \, d^\alpha}{s^2+b^2} \end{aligned}$$

$$
\begin{aligned}
& + \int_{d<|s|<M} \left|\psi'_{2,\varepsilon}(s+x) - \psi'_{1,\varepsilon}(x)\right| \frac{d\left(|\psi'_{2,\varepsilon}(s+\hat{x})| \ + \ |\psi'_{1,\varepsilon}(\hat{x})|\right)}{s^2+b^2} \\
& + \int_{d<|s|<M} \left(||\psi'_{2,\varepsilon}||_{0,\alpha} + ||\psi'_{1,\varepsilon}||_{0,\alpha}\right)^2 \ \frac{d^{\alpha+1}}{s^2+b^2} \\
=: \ & V_1 \ + \ V_2 \ + \ V_3 \ + \ V_4 \ + \ V_5.
\end{aligned}
$$

The first term can be estimated by

$$
\begin{aligned}
V_1 \ & \leq \ C \ (||\psi'_{1,\varepsilon}||_{0,\alpha} + ||\psi'_{2,\varepsilon}||_{0,\alpha})^2 \ d^{1+\alpha} \int_{d<s<M} s^{-2} \\
& \leq \ C \ (||\psi'_{1,\varepsilon}||_{0,\alpha} + ||\psi'_{2,\varepsilon}||_{0,\alpha})^2 \ d^{\alpha}.
\end{aligned}
$$

We easily obtain a similar estimate for V_3 and V_5.

To control V_2, we use once again propositions 2 and 3

$$
\begin{aligned}
V_2 \ & \leq \ C \int_{d<s<M} \frac{d\left(|\psi'_{2,\varepsilon}(s+\hat{x})|^2 \ + \ |\psi'_{1,\varepsilon}(\hat{x})|^2\right)}{s^2+\hat{b}^2} \\
& \leq \ C \ d \int_{d<s<M} \frac{\max(|\psi_{2,\varepsilon}(s+\hat{x})|, |\psi_{1,\varepsilon}(\hat{x})|)^{\frac{2\alpha}{1+\alpha}}}{s^{1+\mu} \ \hat{b}^{1-\mu}}.
\end{aligned}
$$

Choosing $1-\mu = \alpha$ yields

$$
\begin{aligned}
\frac{\max(|\psi_{2,\varepsilon}(s+\hat{x})|, |\psi_{1,\varepsilon}(\hat{x})|)^{\frac{2\alpha}{1+\alpha}}}{s^{1+\mu} \ \hat{b}^{1-\mu}} \ & \leq \ \max(|\psi_{2,\varepsilon}(s+\hat{x})|, |\psi_{1,\varepsilon}(\hat{x})|)^{[\frac{2\alpha}{1+\alpha}-(1-\mu)]} \ s^{\alpha-2} \\
& \leq \ (||\psi_{1,\varepsilon}||_{0,\alpha} + ||\psi_{2,\varepsilon}||_{0,\alpha})^{\frac{\alpha(1-\alpha)}{1+\alpha}} \ s^{\alpha-2}.
\end{aligned}
$$

From (24), it follows that $V_2 \ \leq \ C(\varepsilon) \ d^{\alpha}$. The same argument applies to V_4. In summary we conclude that

$$
|U_2| \ \leq \ C(\varepsilon) \ d^{\alpha}, \tag{72}
$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in δ.

Estimates for U_3 and U_4:
Both terms can be treated in the same fashion. We only present the case of U_3. Using the fact that $\int_{|s|<d} \frac{s}{s^2+a^2} = 0$, we have

$$
\begin{aligned}
U_3 \ = \ & \int_{|s|<d} \frac{s[\psi'_{2,\varepsilon}(s+x) - \psi'_{1,\varepsilon}(x)]}{s^2+b^2} \\
= \ & \int_{|s|<d} \frac{s[\psi'_{2,\varepsilon}(s+x) - \psi'_{2,\varepsilon}(x)]}{s^2+b^2} \ + \ [\psi'_{2,\varepsilon}(x) - \psi'_{1,\varepsilon}(x)] \int_{|s|<d} \left(\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right) \\
=: \ & W_1 + W_2.
\end{aligned}
$$

The first term can be estimated by

$$|W_1| \;\leq\; C \int_{0<s<d} \frac{s^{1+\alpha}||\psi'_{2,\varepsilon}||_{0,\alpha}}{s^2} \;\leq\; C\; ||\psi'_{2,\varepsilon}||_{0,\alpha}\; d^{\alpha}.$$

As for the other term, we have by (54)-(55)

$$\begin{aligned}
|W_2| \;&\leq\; \left|\psi'_{2,\varepsilon}(x) - \psi'_{1,\varepsilon}(x)\right| \int_{|s|<d} \frac{|b-a|\,|s|\,|a+b|}{(s^2+a^2)(s^2+b^2)} \\
&\leq\; \left|\psi'_{2,\varepsilon}(x) - \psi'_{1,\varepsilon}(x)\right| \int_{|s|<d} \frac{|b-a|}{s^2+a^2} \\
&\quad+ \int_{|s|<d} \left|\psi'_{2,\varepsilon}(s+x) - \psi'_{1,\varepsilon}(x)\right| \frac{|b-a|}{s^2+b^2} \\
&\quad+ \int_{|s|<d} \left|\psi'_{2,\varepsilon}(s+x) - \psi'_{2,\varepsilon}(x)\right| \frac{|b-a|}{s^2+b^2} \\
&\leq\; |\psi'_{2,\varepsilon}(x) - \psi'_{1,\varepsilon}(x)| \int_{0<s<d} \frac{||\psi'_{2,\varepsilon}||_{0,\alpha} s^{1+\alpha} + s|\psi'_{2,\varepsilon}(x)|}{s^2+a^2} \\
&\quad+ \int_{0<s<d} |\psi'_{2,\varepsilon}(s+x) - \psi'_{1,\varepsilon}(x)| \,\frac{||\psi'_{2,\varepsilon}||_{0,\alpha} s^{1+\alpha} + s|\psi'_{2,\varepsilon}(s+x)|}{s^2+b^2} \\
&\quad+ \int_{0<s<d} ||\psi'_{2,\varepsilon}||_{0,\alpha}\, s^{\alpha} \frac{s\,||\psi'_{2,\varepsilon}||_{0,\alpha}}{s^2+b^2} \\
&\leq\; C\; ||\psi'_{2,\varepsilon}||^2_{0,\alpha} \int_{0<s<d} s^{\alpha-1} \\
&\quad+ C \int_{0<s<d} \frac{s\max(|\psi'_{1,\varepsilon}(x)|, |\psi'_{2,\varepsilon}(x)|)^2}{s^2+a^2} \\
&\quad+ C \int_{0<s<d} \frac{s\max(|\psi'_{1,\varepsilon}(x)|, |\psi'_{2,\varepsilon}(s+x)|)^2}{s^2+b^2} \\
&\leq\; C\; ||\psi'_{2,\varepsilon}||^2_{0,\alpha} d^{\alpha} \;+\; C \int_{0<s<d} \frac{s\max(|\psi_{1,\varepsilon}(x)|, |\psi_{2,\varepsilon}(x)|)^{\frac{2\alpha}{1+\alpha}}}{s^{1+\mu}\; a^{1-\mu}} \\
&\quad+ C \int_{0<s<d} \frac{s\max(|\psi_{1,\varepsilon}(x)|, |\psi_{2,\varepsilon}(s+x)|)^{\frac{2\alpha}{1+\alpha}}}{s^{1+\mu}\; b^{1-\mu}}.
\end{aligned}$$

Choosing again $1-\mu = \alpha$ yields

$$|W_2| \;\leq\; C \left(||\psi'_{2,\varepsilon}||^2_{0,\alpha} \;+\; \max(||\psi_{1,\varepsilon}||_{0,\alpha}, ||\psi_{2,\varepsilon}||_{0,\alpha})^{\frac{\alpha(1-\alpha)}{1+\alpha}} \right) d^{\alpha}.$$

It follows that

$$|U_3| \;\leq\; C(\varepsilon) d^{\alpha}, \tag{73}$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in δ. By a combination of (67)–(73) it now follows that

$$|R_3| \;\leq\; C(\varepsilon)\; d^{\alpha}, \tag{74}$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, unifomly in δ.

B.3.4 End of the proof of Lemma 6: Hölder continuity of $J^{\varepsilon,\delta}$ for $\delta > 0$

Collecting the estimates (59), (61), (62), (66) and (74), we obtain that for any $\alpha < \alpha_0$, for any $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$, and for any $X, \hat{X} \in \Gamma_1$, $|X|, |\hat{X}| \le R_0$,

$$\left\{ \begin{array}{lll} |J_{\varepsilon,\delta}\varphi(X)| & \le & C(\varepsilon) \\ |J_{\varepsilon,\delta}(X) - J_{\varepsilon,\delta}(\hat{X})| & \le & C(\varepsilon)\, d^\alpha, \end{array} \right.$$

where $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$, uniformly with respect to δ. Lemma 6, for $\delta > 0$ follows immediately.

B.4 Lemma 6, the case $\delta = 0$

Recall that for $X \in \Gamma_1 \cap B(0, R_0)$, $J^{\varepsilon,0}\varphi$ has the form

$$J^{\varepsilon,0}\varphi(X) \quad = \quad \left\{ \begin{array}{l} \int_{|s|<M} \left(\frac{b_0 - s\psi'_{1,\varepsilon}(x)}{s^2+b_0^2} - \frac{a_0 - s\psi'_{1,\varepsilon}(x)}{s^2+a_0^2} \right) \phi(s+x) \quad \text{if } X \ne 0 \\ \int_{|s|<M} \frac{\psi_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2} \phi(s) \quad \text{if } X = 0, \end{array} \right.$$

where $a_0 = \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x), b_0 = \psi_{2,\varepsilon}(s+x) - \psi_{1,\varepsilon}(x)$. When $X \ne 0$, the estimates of section B.2 remain valid, since the denominators of the kernel are always greater than $|\psi_{1,\varepsilon}(x)| > 0$. When $X = 0$, we have

$$\begin{array}{lll} |J^{\varepsilon,0}\varphi(0)| & \le & \displaystyle\int_{|s|<M} \frac{||\psi_{2,\varepsilon}||_{1,\alpha} |s|^{1+\alpha}}{s^2} ||\phi||_\infty \\ & \le & ||\psi_{2,\varepsilon}||_{1,\alpha}\, ||\phi||_\infty\, M^\alpha \;\le\; C\,\varepsilon^\nu\, ||\phi||_\infty . \end{array}$$

It follows that

$$||J^{\varepsilon,0}\varphi||_{L^\infty(\Gamma_1 \cap B(0,R_0))} \quad \le \quad C(\varepsilon) ||\varphi||_{0,\alpha},$$

where $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$.

As for Hölder estimates, when both X and $\hat{X}$ are different from 0, the quantities $|J^{\varepsilon,0}\varphi(X) - J^{\varepsilon,0}\varphi(\hat{X})|$ can be estimated exactly as in B.3, again because the denominators of all the kernels involved in these estimates never vanish. Therefore, we only need to examine $|J^{\varepsilon,0}\varphi(X) - J^{\varepsilon,0}\varphi(0)|$ when $X \ne 0$.

A careful analysis of the previous estimates shows that only the term S_2 in section B.3.2 and $T_1 + T_2$ in section B.3.3 require a modified treatment compared to the case $\delta > 0$. Indeed, we only used the fact that the denominators in the integrands are greater than $s^2$to control the terms S_3, S_4 and T_3.

The term S_2: When $\hat{X}=0$, this term reduces to (see (63))

$$S_2 \;=\; -\int_{|s|<d} \frac{\psi_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2}\,[\phi(s)-\phi(0)]\,,$$

(here $d=|x|$) which can be bounded by

$$\begin{aligned}|S_2| &\leq ||\phi||_{0,\alpha}\int_{|s|<d}\frac{s^\alpha|\psi_{2,\varepsilon}(s)-\psi_{2,\varepsilon}(0)|}{s^2+\psi_{2,\varepsilon}(s)^2}\\ &\leq ||\phi||_{0,\alpha}\,||\psi_{2,\varepsilon}||_{1,0}\int_{|s|<d}s^{\alpha-1}\\ &\leq C(\varepsilon)\,||\phi||_{0,\alpha}\,d^\alpha.\end{aligned}$$

The term T_1+T_2: When $\hat{X}=0$, this expression reduces to

$$T_1+T_2 \;=\; \int_{|s|<M}\frac{b_0-s\psi'_{2,\varepsilon}(s+x)}{s^2+b_0^2}-\frac{\psi_{2,\varepsilon}(s)-s\psi'_{2,\varepsilon}(s)}{s^2+\psi^2_{2,\varepsilon}(s)}-\frac{a_0}{s^2+a_0^2}.$$

Note that since $\psi_{2,\varepsilon}(s)=O(s^{1+\alpha})$, and $\psi'_{2,\varepsilon}(s)=O(s^\alpha)$, the second term is integrable with an integral equal to

$$\begin{aligned}&\lim_{\rho\to 0}\int_\rho^M\frac{\psi_{2,\varepsilon}(s)-s\psi'_{2,\varepsilon}(s)}{s^2+\psi^2_{2,\varepsilon}(s)}+\int_{-M}^{-\rho}\frac{\psi_{2,\varepsilon}(s)-s\psi'_{2,\varepsilon}(s)}{s^2+\psi^2_{2,\varepsilon}(s)}\\ =&\lim_{\rho\to0}\left[\arctan(\frac{M}{\psi_{2,\varepsilon}(M)})-\arctan(\frac{\rho}{\psi_{2,\varepsilon}(\rho)})\;-\;\arctan(\frac{-M}{\psi_{2,\varepsilon}(-M)})+\arctan(\frac{-\rho}{\psi_{2,\varepsilon}(-\rho)})\right]\\ =&\arctan(\frac{M}{\psi_{2,\varepsilon}(M)})-\arctan(\frac{-M}{\psi_{2,\varepsilon}(-M)})-\pi.\end{aligned}$$

It now follows that

$$\begin{aligned}T_1+T_2 \;=\;& \left[\arctan(\frac{M}{b_0(x,M)})-\arctan(\frac{M}{\psi_{2,\varepsilon}(M)})\right]\\ &-\left[\arctan(\frac{-M}{b_0(x,-M)})-\arctan(\frac{-M}{\psi_{2,\varepsilon}(-M)})\right]\\ &-\left[\arctan(\frac{M}{a_0})-\arctan(\frac{-M}{a_0})\right]\;+\;\pi.\end{aligned}$$

Arguing as in (68), the absolute value of first two terms are easily bounded by $C(\varepsilon)|x|$. As for the last terms, one has

$$\left|\mp\arctan(\frac{\pm M}{a_0})+\frac{\pi}{2}\right| \;\leq\; \frac{a_0}{M}\;\leq\; C\,||\psi_{2,\varepsilon}-\psi_{1,\varepsilon}||_{1,\alpha}\,|x|^{1+\alpha}.$$

It follows that $|T_1+T_2|\leq C(\varepsilon)\,|x|$, and Lemma 6 also holds in the case $\delta=0$.

C Proof of Lemma 7

We first note that due to (25), and since $Supp\chi \subset B(0,R_0)$, $||\chi||_\infty \leq 1, ||\chi'||_\infty \leq \varepsilon_0$ and $||\psi'_{1,\varepsilon}||_{0,\alpha} \leq \varepsilon^\nu$ (which can be made smaller than ε_0 by taking ε sufficiently small), we only need to show that

$$\begin{aligned} &\max \left[\sup_{|X|\leq R_0} \left| I_2^{\varepsilon,\delta}\varphi(X) \right|, \sup_{|X|,|\hat{X}|\leq R_0} \frac{\left| I_2^{\varepsilon,\delta}\varphi(X) - I_2^{\varepsilon,\delta}\varphi(\hat{X}) \right|}{|X-\hat{X}|^\alpha} \right] \\ &\leq \pi(1+C(\varepsilon))\, ||\phi||_{0,\alpha}. \end{aligned} \tag{75}$$

C.1 The case $\delta > 0$

For $X, \hat{X} \in \Gamma_1 \cap B(0,R_0)$ with abcissae x and $\hat{x}$, we split the expression of $I_2^{\varepsilon,\delta}$ as $\mathcal{I}_2 - \mathcal{I}_1$ with

$$\begin{aligned} \mathcal{I}_1(x) &= \int_{-R_0}^{R_0} \frac{\psi'_{1,\varepsilon}(x)(y-x)}{(x-y)^2 + (\delta+\psi_{2,\varepsilon}(x)-\psi_{1,\varepsilon}(x))^2}\phi(y)\,dy, \\ \mathcal{I}_2(x) &= \int_{-R_0}^{R_0} \frac{(\delta+\psi_{2,\varepsilon}(x)-\psi_{1,\varepsilon}(x))}{(x-y)^2 + (\delta+\psi_{2,\varepsilon}(x)-\psi_{1,\varepsilon}(x))^2}\phi(y)\,dy, \end{aligned}$$

and next estimate the L^∞ and Hölder semi–norm of these two operators.

L^∞ estimate of $\mathcal{I}_1$: Since ϕ has compact support, changing variables to $s = y - x$ yields

$$|\mathcal{I}_1(x)| = \left| \int_{I\!R} \frac{s\psi'_{1,\varepsilon}(x)}{s^2+a^2}\phi(s+x)\,ds \right|.$$

Noting that $\frac{s}{s^2+a^2}$ is an odd function of s, we see that

$$\begin{aligned} |\mathcal{I}_1(x)| &= \left| \int_{I\!R} \frac{s\psi'_{1,\varepsilon}(x)}{s^2+a^2}(\phi(s+x)-\phi(x))\,ds \right| \\ &\leq ||\psi'_{1,\varepsilon}||_\infty \int_{0<s\leq M} \frac{s^{1+\alpha}||\phi||_{0,\alpha}}{s^2+a^2}\,ds \\ &\leq C\,\varepsilon^\nu\,||\phi||_{0,\alpha}, \end{aligned} \tag{76}$$

where C only depends on ε_0, M and ψ_1, but is independent on ε and δ.

Hölder estimate of $\mathcal{I}_1$: We form

$$\begin{aligned} \mathcal{I}_1(x) - \mathcal{I}_1(\hat{x}) &= \int_{I\!R} \frac{s[\psi'_{1,\varepsilon}(x) - \psi'_{1,\varepsilon}(\hat{x})]}{s^2+a^2}\,\phi(s+x)\,ds \\ &+ \psi'_{1,\varepsilon}(\hat{x}) \int_{I\!R} \left(\frac{s}{s^2+a^2} - \frac{s+d}{(s+d)^2+\hat{a}^2} \right) \phi(s+x)\,ds, \end{aligned} \tag{77}$$

with $d := x - \hat{x}$. The first integral above is easily estimated using (24) and the fact that $s \to \frac{s}{s^2+a^2}$ is an odd function: it is bounded by

$$\begin{aligned}
&||\psi'_{1,\varepsilon}||_{0,\alpha}\,|x-\hat{x}|^\alpha \left| \int_{\mathbb{R}} \frac{s}{s^2+a^2}[\phi(s+x)-\phi(x)]\,ds \right| \\
\leq\ & ||\psi'_{1,\varepsilon}||_{0,\alpha}\,|x-\hat{x}|^\alpha ||\phi||_{0,\alpha} \int_{|s|<M} \frac{s^{1+\alpha}}{s^2+a^2}\,ds \\
\leq\ & C\,\varepsilon^\nu\,||\phi||_{0,\alpha}\,|x-\hat{x}|^\alpha, \quad \nu = \alpha_0 - \alpha. \qquad (78)
\end{aligned}$$

To treat the second term, let us assume (without loss of generality) that $d = x - \hat{x} \geq 0$ and rewrite the integral factor in this term as

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{s}{s^2+a^2} - \frac{s+d}{(s+d)^2+\hat{a}^2} \right) \phi(s+x)\,ds \\
=\ & \int_{|s|<4d} \frac{s}{s^2+a^2}[\phi(s+x)-\phi(x)] \;-\; \int_{|s|<4d} \frac{s+d}{(s+d)^2+\hat{a}^2}[\phi(s+x)-\phi(x-d)]\,ds \\
& - \int_{|s|<4d} \frac{s+d}{(s+d)^2+\hat{a}^2}[\phi(x-d)-\phi(x)]\,ds \\
& + \int_{|s|>4d} \left(\frac{s}{s^2+a^2} - \frac{s+d}{(s+d)^2+\hat{a}^2} \right) [\phi(s+x)-\phi(x)] \\
=\ & i_1 + i_2 + i_3 + i_4.
\end{aligned}$$

Here we have used the fact that $\frac{s}{s^2+a^2}$ and $\frac{s+d}{(s+d)^2+\hat{a}^2}$ are odd functions of s and $s+d$ respectively. We estimate i_1 by

$$\begin{aligned}
|i_1| &= \left| \int_{|s|<4d} \frac{s}{s^2+a^2}[\phi(s+x)-\phi(x)]\,ds \right| \\
&\leq ||\phi||_{0,\alpha} \int_{|s|<4d} \frac{s^{1+\alpha}}{s^2+a^2}\,ds \;\leq\; C\,||\phi||_{0,\alpha}\,d^\alpha.
\end{aligned}$$

The second term i_2 can be estimated in the same way, as $|\phi(s+x)-\phi(x-d)| \leq ||\phi||_{0,\alpha}(s+d)^\alpha$. For i_3 we have

$$\begin{aligned}
|i_3| &= |\phi(x-d)-\phi(x)| \left| \int_{|s|<4d} \frac{s+d}{(s+d)^2+\hat{a}^2}\,ds \right| \\
&\leq ||\phi||_{0,\alpha}\,d^\alpha \left| \int_{-3d}^{5d} \frac{\sigma}{\sigma^2+\hat{a}^2}\,d\sigma \right| \;\leq\; ||\phi||_{0,\alpha}\,d^\alpha \int_{3d}^{5d} \frac{\sigma}{\sigma^2+\hat{a}^2}\,d\sigma \\
&= ||\phi||_{0,\alpha}\,d^\alpha\; 1/2 \ln\!\left(\frac{25d^2+\hat{a}^2}{9d^2+\hat{a}^2}\right) \;\leq\; \ln\!\left(\frac{5}{3}\right) ||\phi||_{0,\alpha}\,d^\alpha.
\end{aligned}$$

Finally, the remaining term, i_4, can be bounded as follows:

$$
\begin{aligned}
|i_4| &= \left| \int_{|s|>4d} \left(\frac{s}{s^2+a^2} - \frac{s+d}{(s+d)^2+\hat{a}^2} \right) [\phi(s+x)-\phi(x)]\, ds \right| \\
&= \left| \int_{|s|>4d} \frac{s^2 d + s d^2 + s(\hat{a}-a)(a+\hat{a}) - a^2 d}{(s^2+a^2)\left((s+d)^2+\hat{a}^2\right)} [\phi(s+x)-\phi(x)]\, ds \right| \\
&\leq C \int_{|s|>4d} \left(\frac{d}{s^2} + \frac{d^2}{s^3} + \frac{|a-\hat{a}|}{s^2} \right) |s|^\alpha \, ||\phi||_{0,\alpha}\, ds \\
&\leq C \int_{|s|>4d} \left(\frac{d}{s^2} + \frac{d^2}{s^3} + \frac{d||\psi'_{2,\varepsilon} - \psi'_{1,\varepsilon}||_\infty}{s^2} \right) |s|^\alpha \, ||\phi||_{0,\alpha}\, ds \\
&\leq C\, ||\phi||_{0,\alpha}\, d^\alpha .
\end{aligned}
$$

It follows that the second term in (77) can be bounded by $C\,||\psi'_{1,\varepsilon}||_\infty \,||\phi||_{0,\alpha}\, |x-\hat{x}|^\alpha$. In combination with (78), we conclude that

$$
|\mathcal{I}_1(x) - \mathcal{I}_1(\hat{x})| \;\; \leq \;\; C\, \varepsilon^\nu \, ||\phi||_{0,\alpha}\, |x-\hat{x}|^\alpha . \tag{79}
$$

L^∞ **estimate of** I_2**:** Changing variables from y to $s = y - x$, and then to $t = s/a$, which is well defined since $a = (\delta + \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x)) \geq \delta > 0$, we easily see that

$$
\begin{aligned}
|\mathcal{I}_2(x)| &= \left| \int_{I\!R} \frac{a}{s^2+a^2}\, \phi(s+x)\, ds \right| \\
&= ||\phi||_\infty \int_{I\!R} \frac{1}{1+t^2}\, dt \\
&= \pi\, ||\phi||_\infty .
\end{aligned} \tag{80}
$$

Hölder estimate of I_2**:** Let $X, \hat{X} \in \Gamma_1 \cap B(0, R_0)$, with respective abcissae x and $\hat{x}$. We form

$$
\begin{aligned}
\mathcal{I}_2(x) - \mathcal{I}_2(\hat{x}) &= \int_{I\!R} \frac{a}{s^2+a^2} \phi(s+x)\, ds - \int_{I\!R} \frac{\hat{a}}{s^2+\hat{a}^2} \phi(s+\hat{x})\, ds \\
&= \int_{I\!R} \frac{1}{1+t^2} \phi(at+x)\, dt - \int_{I\!R} \frac{1}{1+t^2} \phi(\hat{a}t+\hat{x})\, dt .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|\mathcal{I}_2(x) - \mathcal{I}_2(\hat{x})| &\leq \int_{I\!R} \frac{1}{1+t^2} |\phi(at+x) - \phi(\hat{a}t+\hat{x})|\, dt \\
&\leq ||\phi||_{0,\alpha}\, |x-\hat{x}|^\alpha \int_{I\!R} \frac{1}{1+t^2} \left| \frac{a-\hat{a}}{x-\hat{x}} t + 1 \right|^\alpha dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; ||\phi||_{0,\alpha}\,|x-\hat{x}|^\alpha \int_{I\!R} \frac{1}{1+t^2} \left| \frac{(\psi_{2,\varepsilon}-\psi_{1,\varepsilon})(x)-(\psi_{2,\varepsilon}-\psi_{1,\varepsilon})(\hat{x})}{x-\hat{x}} t+1 \right|^\alpha dt \\
&\leq \; ||\phi||_{0,\alpha}\,|x-\hat{x}|^\alpha \int_{I\!R} \frac{1}{1+t^2} (C\,\varepsilon^\nu\,|t|+1)^\alpha\,dt.
\end{aligned}
$$

By the Lebesgue dominated convergence Theorem, the last integral converges to $\int_{I\!R} \frac{1}{1+t^2}\,dt = \pi$ as $\varepsilon \to 0$. It follows that

$$
|\mathcal{I}_2(x)-\mathcal{I}_2(\hat{x})| \;\leq\; (\pi + C(\varepsilon))\,||\phi||_{0,\alpha}\,|x-\hat{x}|^\alpha, \tag{81}
$$

with $C(\varepsilon) \to 0$, as $\varepsilon \to 0$.

The estimate (75) follows from a combination of (76), (79), (80), and (81). This completes the proof of Lemma 7 in the case $\delta > 0$.

C.2 The case $\delta = 0$

We remark that the estimates of $\mathcal{I}_1(x), \mathcal{I}_2(x), \mathcal{I}_1(x)-\mathcal{I}_1(\hat{x}), \mathcal{I}_2(x)-\mathcal{I}_2(\hat{x})$ of section C.1 remain valid when $\delta = 0$ if both $X \neq 0$ and $\hat{X} \neq 0$, since in this case, the denominators of the kernels do not vanish. Thus, to establish the lemma when $\delta = 0$, we only need to check that $|I^{\varepsilon,0}\varphi(0)| \leq (\pi + C(\varepsilon))||\phi||_{0,\alpha}$ and that $|I^{\varepsilon,0}\varphi(X) - I^{\varepsilon,0}\varphi(0)| \leq (\pi + C(\varepsilon))||\phi||_{0,\alpha}\,|X|^\alpha$, with $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$. The first inquality is a straightforward consequence of the definition of $I^{\varepsilon,0}\varphi(0)$ and of the fact that $\phi(0) = \varphi(0)$.

To prove the second estimate, we form

$$
\begin{aligned}
|I^{\varepsilon,0}\varphi(X) - I^{\varepsilon,0}\varphi(0)| \;&\leq\; \left| \int_{|s|<M} \frac{a_0}{s^2+a_0^2}\phi(s+x) - \pi\phi(0) \right| \\
&\quad + \left| \int_{|s|<M} \frac{s\psi'_{1,\varepsilon}(x)}{s^2+a_0^2}\phi(s+x) \right| \\
&\leq\; \left| \int_{I\!R} \frac{1}{1+t^2}\phi(a_0 t + x) - \int_{I\!R} \frac{1}{1+t^2}\phi(0) \right| \\
&\quad + \left| \int_{|s|<M} \frac{s\psi'_{1,\varepsilon}(x)}{s^2+a_0^2}[\phi(s+x)-\phi(x)] \right|
\end{aligned}
$$

The $\mathcal{C}^{0,\alpha}$ regularity of $\psi'_{1,\varepsilon}$ implies that the second term can be estimated by

$$
\begin{aligned}
|\psi'_{1,\varepsilon}(x)|\,||\phi||_{0,\alpha} \int_{|s|<M} \frac{s^{1+\alpha}}{s^2+a_0^2} \;&\leq\; C||\psi_{1,\varepsilon}||_{1,\alpha}\,|x|^\alpha\,||\phi||_{0,\alpha} \\
&\leq\; C\,\varepsilon^\nu\,||\phi||_{0,\alpha}\,|X|^\alpha.
\end{aligned}
$$

As for the first term, we write it as

$$\begin{aligned}\left|\int_{I\!R} \frac{1}{1+t^2}\left[\phi(a_0 t + x) - \phi(0)\right]\right| &\leq ||\phi||_{0,\alpha} \int_{I\!R} \frac{1}{1+t^2}\left|\frac{\psi_{2,\varepsilon}(x)-\psi_{1,\varepsilon}(x)}{x} t + 1\right|^{\alpha} |x|^{\alpha} \\ &\leq |X|^{\alpha}\,||\phi||_{0,\alpha} \int_{I\!R} \frac{1}{1+t^2} |C\varepsilon^{\nu} t + 1|^{\alpha}\end{aligned}$$

It easily follows from the Lebesgue dominated convergence theorem that the integral above (which is independent of x) converges to π as $\varepsilon \to 0$. Combining the two previous estimates we obtain

$$|I^{\varepsilon,\delta}\varphi(X) - I^{\varepsilon,\delta}\varphi(0)| \quad \leq \quad (\pi + C(\varepsilon))\,||\phi||_{0,\alpha}\,|X|^{\alpha},$$

with $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$, as desired. ∎

D Proof of Lemma 8

In this section, we show that for fixed ε,

$$\forall\, \varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2), \quad \lim_{\delta\to 0} K^{\varepsilon,\delta}\varphi \quad = \quad K^{\varepsilon,0}\varphi \quad \text{in } \mathcal{C}^{0,\alpha}(\Gamma_1).$$

We then show a similar result for $I^{\varepsilon,\delta}$, $J^{\varepsilon,\delta}$, but it is not as strong: For these operators, we are only able to show pointwise convergence in $\mathcal{C}^{0,\alpha'}(\Gamma_1)$ for all $0 < \alpha' < \alpha$, when $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$.

The case of $K^{\varepsilon,\delta}$ is the easiest. Since $\Gamma_2 \cap \{|x| \leq \varepsilon\} = \Gamma_{2,\varepsilon} \cap \{|x| \leq \varepsilon\}$ the denominators are bounded away from 0 in the expression (26) (this also holds for $\delta = 0$). Hence, $K^{\varepsilon,\delta}$ is an integral operator with a $\mathcal{C}^{1+\alpha}$ kernel, and one can take limits in the integrand to obtain

$$\lim_{\delta\to 0} K^{\varepsilon,\delta}\varphi \quad = \quad K^{\varepsilon,0}\varphi,$$

in the sense of $C^{0,\alpha}(\Gamma_1)$.

Let $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$ and $0 < \alpha' < \alpha$. Assume that $\chi I^{\varepsilon,\delta}\varphi$ does not converge to $\chi I^{\varepsilon,0}\varphi$ in $\mathcal{C}^{0,\alpha'}(\Gamma_1)$. Then for some $\rho > 0$ there is a sequence which satisfies

$$||\chi I^{\varepsilon,\delta_n}\varphi - \chi I^{\varepsilon,0}\varphi||_{0,\alpha'} \quad > \quad \rho. \tag{82}$$

Lemma 7 implies that $\chi I^{\varepsilon,\delta_n}\varphi$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Gamma_1)$. Since $\mathcal{C}^{0,\alpha}(\Gamma_1)$ is compactly embedded in $\mathcal{C}^{0,\alpha'}(\Gamma_1)$, we may assume, after extraction of a subsequence, that $(\chi I^{\varepsilon,\delta_n}\varphi)$ converges to some function $\xi \in \mathcal{C}^{0,\alpha'}(\Gamma_1)$. We show in Proposition 5 below that $\chi I^{\varepsilon,\delta}\varphi$ converges uniformly to $\chi I^{\varepsilon,0}\varphi$. Uniqueness of the limit implies that

$\xi \equiv \chi I^{\varepsilon,0}\varphi$, which contradicts (82), and proves the statement of Lemma 8 concerning $\chi I^{\varepsilon,\delta}$.

Using Lemma 6, the same argument shows that $\chi J^{\varepsilon,\delta}\varphi$ converges to $\chi J^{\varepsilon,0}\varphi$ in $\mathcal{C}^{0,\alpha'}(\Gamma_1)$. Here, we use Proposition 4 below which shows that $J^{\varepsilon,\delta}\varphi(X)$ converges pointwise to $J^{\varepsilon,0}\varphi(X)$, for $X \in \Gamma_1 \cap B(0,R_0)$.

D.1 Pointwise convergence of $J^{\varepsilon,\delta}\varphi(X)$ as $\delta \to 0$

We prove the following:

Proposition 4 *Let $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$. Then for any $\varepsilon \leq \varepsilon_0/2$ and any $X \in \Gamma_1 \cap B(0,R_0)$, $lim_{\delta\to 0} J^{\varepsilon,\delta}(X) = J^{\varepsilon,0}\varphi(X)$.*

Proof: Recall that a_0 and b_0 denote the quantities

$$\begin{aligned} a_0 &= \psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x) \\ b_0 &= \psi_{2,\varepsilon}(s+x) - \psi_{1,\varepsilon}(x). \end{aligned}$$

For $X \in \Gamma_1 \cap B(0,R_0), X \neq 0$, the kernel

$$j_{\varepsilon,\delta}(s,x) = \frac{b - s\psi'_{1,\varepsilon}(x)}{s^2+b^2} - \frac{a - s\psi'_{1,\varepsilon}(x)}{s^2+a^2},$$

converges as $\delta \to 0$ a.e. $s \in (-M,M)$, to

$$\begin{aligned} &\frac{\psi_{2,\varepsilon}(s+x) - \psi_{1,\varepsilon}(x) - s\psi'_{1,\varepsilon}(x)}{s^2 + (\psi_{2,\varepsilon}(s+x) - \psi_{1,\varepsilon}(x))^2} - \frac{\psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x) - s\psi'_1(x)}{s^2 + (\psi_{2,\varepsilon}(x) - \psi_{1,\varepsilon}(x))^2} \\ &= \frac{b_0 - s\psi'_{1,\varepsilon}(x)}{s^2+b_0^2} - \frac{a_0 - s\psi'_{1,\varepsilon}(x)}{s^2+a_0^2}. \end{aligned}$$

Since we also have

$$|j_{\varepsilon,\delta}(s,x)| \leq \frac{|b - s\psi'_{1,\varepsilon}(x)|}{s^2 + \psi_{1,\varepsilon}(x)^2} + \frac{|a - s\psi'_{1,\varepsilon}(x)|}{s^2 + \psi_{1,\varepsilon}(x)^2},$$

which is integrable on $(-M,M)$, the Lebesgue dominated convergence theorem implies that, for any $X \in \Gamma_1 \cap B(0,R_0), X \neq 0$,

$$\begin{aligned} \lim_{\delta\to 0} J^{\varepsilon,\delta}\varphi(X) &= \int_{|s|<M} \left(\frac{b_0 - s\psi'_{1,\varepsilon}(x)}{s^2+b_0^2} - \frac{a_0 - s\psi'_{1,\varepsilon}(x)}{s^2+a_0^2} \right) \phi(s+x) \\ &= J^{\varepsilon,0}\varphi(X). \end{aligned}$$

When $X = 0$, the expression of $J_{\varepsilon,\delta}\varphi(X)$ reduces to

$$\begin{aligned}
J^{\varepsilon,\delta}\varphi(0) &= \int_{|s|<M} \left(\frac{\psi_{2,\varepsilon}(s)+\delta}{s^2+(\psi_{2,\varepsilon}(s)+\delta)^2} - \frac{\delta}{s^2+\delta^2} \right) \phi(s) \\
&= \int_{|s|<M} \left(\frac{\psi_{2,\varepsilon}(s)+\delta}{s^2+(\psi_{2,\varepsilon}(s)+\delta)^2} - \frac{\delta}{s^2+\delta^2} \right) [\phi(s)-\phi(0)] \\
&\quad + \phi(0) \int_{|s|<M} \frac{\psi_{2,\varepsilon}(s)+\delta - s\psi'_{2,\varepsilon}(s)}{s^2+(\psi_{2,\varepsilon}(s)+\delta)^2} \\
&\quad + \phi(0) \int_{|s|<M} \frac{s\psi'_{2,\varepsilon}(s)}{s^2+(\psi_{2,\varepsilon}(s)+\delta)^2} \\
&\quad - \phi(0) \int_{|s|<M} \frac{\delta}{s^2+\delta^2} \\
&=: T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

The term T_2 can be integrated explicitely to obtain

$$\begin{aligned}
T_2 &= \phi(0)\ [\arctan(\frac{s}{\psi_{2,\varepsilon}(s)+\delta})]_{-M}^{M} \\
&\to \phi(0)\ [\arctan(\frac{M}{\psi_{2,\varepsilon}(M)}) - \arctan(\frac{-M}{\psi_{2,\varepsilon}(-M)})], \quad \text{as } \delta \to 0.
\end{aligned}$$

We remark that since $\psi_{2,\varepsilon}(s) = O(|s|^{1+\alpha})$,

$$\begin{aligned}
\lim_{\rho\to 0^\pm} \int_{\rho}^{\pm M} \frac{\psi_{2,\varepsilon}(s) - s\psi'_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}^2(s)} &= \arctan(\frac{\pm M}{\psi_{2,\varepsilon}(\pm M)}) - \lim_{\rho\to 0^\pm} \arctan(\frac{\rho}{\psi_{2,\varepsilon}(\rho)}) \\
&= \arctan(\frac{\pm M}{\psi_{2,\varepsilon}(\pm M)}) \mp \pi/2.
\end{aligned}$$

It follows that

$$\lim_{\delta\to 0} T_2 = \phi(0) \int_{-M}^{M} \frac{\psi_{2,\varepsilon}(s) - s\psi'_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}^2(s)} + \pi\phi(0). \tag{83}$$

It is easily checked that the integrands in T_1 and T_3 converge a.e. $s \in (-M, M)$ to the corresponding expression with $\delta = 0$. Furthermore, the integrand in T_1 is bounded by

$$\left(\frac{\psi_{2,\varepsilon}(s)+\delta}{2|s|\ (\psi_{2,\varepsilon}(s)+\delta)} + \frac{\delta}{2|s|\ \delta} \right) ||\phi||_{0,\alpha}\, |s|^{\alpha} \leq ||\phi||_{0,\alpha}\, |s|^{\alpha-1},$$

which is integrable on $(-M, M)$. The integrand in T_3 can be bounded using propositions 2 and 3

$$\left| \frac{s\psi'_{2,\varepsilon}(s)}{s^2+(\psi_{2,\varepsilon}(s)+\delta)^2} \right| \leq \frac{C|s||\psi_{2,\varepsilon}(s)|^{\frac{\alpha}{1+\alpha}}}{|s|^{1+\mu}\, |\psi_{2,\varepsilon}(s)+\delta|^{1-\mu}}.$$

Choosing $1-\mu = \frac{\alpha}{1+\alpha}$, i.e. $0 < \mu = \frac{1}{1+\alpha} < 1$, we see that the above term is smaller than $Cs^{-\mu}$ which is also integrable on $(-M, M)$. An application of the Lebesgue dominated convergence shows that

$$\lim_{\delta\to 0} T_1 = \int_{|s|<M} \frac{\psi_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2}\,[\phi(s)-\phi(0)] \tag{84}$$

$$\lim_{\delta\to 0} T_3 = \phi(0)\int_{|s|<M} \frac{s\psi'_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2}. \tag{85}$$

The term T_4 can be integrated explicitely and

$$\begin{aligned}\lim_{\delta\to 0} T_4 &= -\lim_{\delta\to 0}\phi(0)\,[\arctan(\frac{s}{\delta})]_{-M}^{M}\\ &= -\pi\phi(0).\end{aligned} \tag{86}$$

Gathering (83-86), we see

$$\begin{aligned}\lim_{\delta\to 0} J^{\varepsilon,\delta}\varphi(0) &= \int_{|s|<M} \frac{\psi_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2}\,[\phi(s)-\phi(0)]\\ &\quad + \phi(0)\int_{|s|<M} \frac{\psi_{2,\varepsilon}(s)-s\psi'_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2}\\ &\quad + \phi(0)\int_{|s|<M} \frac{s\psi'_{2,\varepsilon}}{s^2+\psi_{2,\varepsilon}(s)^2}\\ &= \int_{|s|<M} \frac{\psi_{2,\varepsilon}(s)}{s^2+\psi_{2,\varepsilon}(s)^2}\phi(s)\\ &= J^{\varepsilon,0}\varphi(0),\end{aligned}$$

which completes the proof of Proposition 4. ■

D.2 Convergence of $\chi I^{\varepsilon,\delta}\varphi$ in $L^\infty(\Gamma_1)$, as $\delta \to 0$

Proposition 5 *Let $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma_2)$. Then, for any $\varepsilon \le \varepsilon_0/2$, we have*

$$\lim_{\delta\to 0} ||\chi I^{\varepsilon,\delta}\varphi - \chi I^{\varepsilon,0}\varphi||_\infty = 0.$$

Proof: We again split $I^{\varepsilon,\delta}$ in two parts $\mathcal{I}_2^{\varepsilon,\delta}\varphi - \mathcal{I}_1^{\varepsilon,\delta}\varphi$ as in the proof of Lemma 7. In particular, when $X \in \Gamma_1 \cap B(0, R_0)$ and $\delta = 0$,

$$\mathcal{I}_1^{\varepsilon,0}\varphi(X) = \begin{cases} \int_{|s|<M} \frac{s\psi'_{1,\varepsilon}(x)}{s^2+a_0^2}\phi(s+x) & \text{if } X \neq 0 \\ 0 \quad \text{if } X = 0 \end{cases}$$

$$\mathcal{I}_2^{\varepsilon,0}\varphi(X) = \begin{cases} \int_{|s|<M} \frac{a_0}{s^2+a_0^2}\phi(s+x) & \text{if } X \neq 0 \\ \pi\phi(0) \quad \text{if } X = 0 \end{cases}$$

We first examine the convergence of $\mathcal{I}_1^{\varepsilon,\delta}$. For $X \neq 0, X \in \Gamma_1 \cap B(0,R_0)$, since the integrand is an odd function of s, we have

$$\begin{aligned}
\mathcal{I}_1^{\varepsilon,\delta}\varphi(X) - \mathcal{I}_1^{\varepsilon,0}\varphi(X) &= \int_{|s|<M} s\psi'_{1,\varepsilon}(x)\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2}\right)[\phi(s+x)-\phi(x)] \\
&= \int_{|s|<\delta} s\psi'_{1,\varepsilon}(x)\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2}\right)[\phi(s+x)-\phi(x)] \\
&\quad + \int_{\delta<|s|<M} s\psi'_{1,\varepsilon}(x)\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2}\right)[\phi(s+x)-\phi(x)] \\
&=: T_1 + T_2.
\end{aligned}$$

The term T_1 can be estimated by

$$\begin{aligned}
|T_1| &\leq |\psi'_{1,\varepsilon}(x)|\,||\phi||_{0,\alpha} \int_{|s|<\delta} s^{1+\alpha}\left|\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2}\right| \\
&\leq |\psi'_{1,\varepsilon}(x)|\,||\phi||_{0,\alpha} \int_{|s|<\delta} s^{\alpha-1} \\
&\leq C\,|\psi'_{1,\varepsilon}(x)|\,||\varphi||_{0,\alpha}\,\delta^{\alpha}. \qquad (87)
\end{aligned}$$

As for T_2, we have

$$|T_2| \leq ||\phi||_{0,\alpha} \int_{\delta<|s|<M} |\psi'_{1,\varepsilon}(x)|\,|s|^{1+\alpha}\left|\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2}\right|.$$

Applying the mean value theorem, we see that for any $s \in I\!R$,

$$\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2} = \frac{-2(a_0+\theta\delta)\delta}{(s^2+(a_0+\theta\delta)^2)^2},$$

for some $0 \leq \theta \leq 1$, so that

$$|\psi'_{1,\varepsilon}(x)||s|^{1+\alpha}\left|\frac{1}{s^2+a^2} - \frac{1}{s^2+a_0^2}\right| \leq \delta\frac{|s|^{\alpha}|\psi'_{1,\varepsilon}(x)|}{s^2+a_0^2}.$$

Using once again Propositions 2 and 3, we can estimate the above right-hand side by

$$C\,\delta\,\frac{|s|^{\alpha}|\psi_{1,\varepsilon}(x)|^{\frac{\beta}{1+\beta}}}{s^{1+\frac{1}{1+\beta}}|\psi_{1,\varepsilon}(x)|^{1-\frac{1}{1+\beta}}} \leq C\,\delta\,|s|^{\alpha-1-\frac{1}{1+\beta}},$$

for any $\beta < \alpha_0$. Thus, we obtain

$$\begin{aligned}
|T_2| &\leq C\,||\phi||_{0,\alpha} \int_{\delta<|s|<M} \delta\,|s|^{\alpha-1-\frac{1}{1+\beta}} \\
&\leq \delta^{\alpha}\,C\,||\phi||_{0,\alpha} \int_{\delta<|s|<M} |s|^{-\frac{1}{1+\beta}} \\
&\leq C\,||\varphi||_{0,\alpha}\,\delta^{\alpha}.
\end{aligned}$$

The above inequality together with (87), and the fact that $\mathcal{I}_1^{\varepsilon,\delta}\varphi(0) = \mathcal{I}_1^{\varepsilon,0}\varphi(0) = 0$, imply that

$$||\chi\mathcal{I}_1^{\varepsilon,\delta}\varphi - \chi\mathcal{I}_1^{\varepsilon,0}\varphi||_\infty \;\leq\; C\,||\varphi||_{0,\alpha}\,\delta^a. \tag{88}$$

Next, we consider the convergence of $\mathcal{I}_2^{\varepsilon,\delta}$. Assuming $X \neq 0, X \in \Gamma_1 \cap B(0, R_0)$, we have

$$\begin{aligned}|\mathcal{I}_2^{\varepsilon,\delta}\varphi(X) - \mathcal{I}_2^{\varepsilon,0}\varphi(X)| &= \left|\int_{|s|<M}\frac{a}{s^2+a^2}\phi(s+x) - \int_{|s|<M}\frac{a_0}{s^2+a_0^2}\phi(s+x)\right| \\ &= \left|\int_{\mathbb{R}}\frac{1}{1+t^2}\left[\phi(at+x)-\phi(a_0t+x)\right]\right|.\end{aligned}$$

Recalling that $a = a_0 + \delta$, it follows that

$$\begin{aligned}|\mathcal{I}_2^{\varepsilon,\delta}\varphi(X) - \mathcal{I}_2^{\varepsilon,0}\varphi(X)| &\leq ||\phi||_{0,\alpha}\int_{\mathbb{R}}\frac{\delta^\alpha t^\alpha}{1+t^2} \\ &\leq C\,||\varphi||_{0,\alpha}\,\delta^\alpha.\end{aligned} \tag{89}$$

When $X = 0$, we have

$$\begin{aligned}|\mathcal{I}_2^{\varepsilon,\delta}\varphi(0) - \mathcal{I}_2^{\varepsilon,0}\varphi(0)| &= \left|\int_{|s|<M}\frac{\delta}{s^2+\delta^2}\phi(s) \;-\; \pi\phi(0)\right| \\ &= \left|\int_{\mathbb{R}}\frac{1}{1+t^2}\left[\phi(\delta t)-\phi(0)\right]\right| \\ &\leq ||\phi||_{0,\alpha}\int_{\mathbb{R}}\frac{\delta^\alpha t^\alpha}{1+t^2},\end{aligned}$$

which, in view of (89), shows that

$$||\chi\mathcal{I}_2^{\varepsilon,\delta}\varphi - \chi\mathcal{I}_2^{\varepsilon,0}\varphi||_\infty \;\leq\; C\,||\varphi||_{0,\alpha}\,\delta^\alpha. \tag{90}$$

A combination of (88) and (90) now completes the proof of the proposition. ■

E Proof of lemma 4

In this section we show that the off-diagonal term $L_2^\delta \; : \mathcal{C}^{0,\alpha}(\Gamma_2) \to \mathcal{C}^{0,\alpha}(\Gamma_1)$ is not a compact operator when $\delta = 0$, for any $0 < \alpha < 1$. For simplicity, we only consider the case when Γ_2 is flat around the contact point, i.e., we assume that $\psi_2(y) = 0$ for $|y| < y_0$. Note that in this case D_2 is not strictly convex. The general case can be reduced to the case of a flat Γ_2, by using a decomposition of the operator similar to that of section 2.3. Let $\chi \in \mathcal{C}_c^\infty(-y_0, y_0)$, with $0 \le \chi \le 1$ and $\chi(y) \equiv 1$ for $|y| \le y_0/2$. For $X = (x, \psi_1(x)) \in \Gamma_1$, we write

$$L_2^\delta(\varphi) \;=\; L_2^\delta((1-\chi(|Y|))\varphi) \;+\; L_2^\delta(\chi(|Y|)\varphi).$$

The first operator on the right–hand side has a kernel that remains uniformly bounded with respect to δ, and is thus compact from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1)$ in the limit $\delta = 0$. Setting $\phi(y) = \chi(y)\varphi(y,0)$, the second operator writes for $|X| \le \varepsilon_0$

$$\begin{aligned} R^\delta\varphi(X) \;&:=\; \frac{1}{2\pi}\int_{I\!R} \frac{(\delta - \psi_1(x) - \psi_1'(x)(y-x)}{(x-y)^2 + (\psi_1(x)-\delta)^2}\phi(y)\,dy \\ &=\; \frac{1}{2\pi}\int_{I\!R} \frac{(x-y)\psi_1'(x)}{(x-y)^2 + (\psi_1(x)-\delta)^2}\phi(y)\,dy \;+\; \frac{1}{2\pi}\int_{I\!R} \frac{\phi(t[\delta-\psi_1(x)]+x)}{t^2+1}\,dt, \end{aligned}$$

where we have changed to the variable $t = (y-x)/(\delta - \psi_1(x))$. As $\delta \to 0$, R^δ formally reduces to $R\varphi = 1/2\pi(R_1 + R_2)\varphi$ with

$$\begin{aligned} R_1\varphi(X) \;&=\; \int_{I\!R} \frac{(x-y)\psi_1'(x)}{(x-y)^2 + \psi_1(x)^2}\phi(y)\,dy \\ R_2\varphi(X) \;&=\; \int_{I\!R} \frac{\phi(t|\psi_1(x)| + x)}{t^2+1}\,dt. \end{aligned}$$

It is not difficult to make this convergence argument rigorous, and this shows that $L_2^0 - \frac{1}{2\pi}(R_1 + R_2)$ is $\mathcal{C}^{0,\alpha}$–compact. Proceeding as in Section (C), one can check that both R_1 and R_2 are continuous from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1)$, for any $0 < \alpha < \alpha_0$.

We now show that $R_1 + R_2$ is not compact. We can always assume that the support of the cut–off function χ is sufficiently large to contain $y = 1$. Let $\zeta \in \mathcal{C}_c^1(I\!R)$, such that $\mathrm{Supp}(\zeta) \subset (-y_0, y_0)$, $\zeta(1) \ne 0$ and $\zeta(0) = 0$. For $n \ge 1$ and $Y = (y, 0) \in \Gamma_2$, we define

$$\varphi_n(Y) \;=\; 2^{-n\alpha}\zeta(2^n y).$$

Note that $\chi\varphi_n = \varphi_n$.

Claim 1: The sequence φ_n is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Gamma_2)$.
We first note that since $\zeta \in \mathcal{C}_c^1(I\!R)$, we have for any $0 < \mu < 1$, and any $(y, y') \in \mathrm{supp}(\zeta)^2$,

$$\frac{|\zeta(y) - \zeta(\hat{y})|}{|y-\hat{y}|^\mu} \;\le\; ||\zeta||_1\, |y-\hat{y}|^{1-\mu} \;\le\; ||\zeta||_1\, 2y_0^{1-\mu}.$$

It immediately follows that $\zeta \in \mathcal{C}^{0,\mu}(I\!R)$ with a norm that is bounded by $2y_0^{1-\mu}||\zeta||_1$ for any $0 < \mu < 1$.

Next, $||\varphi_n||_\infty \leq 2^{-n\alpha}||\zeta||_\infty$ tends to 0, while for $y, \hat{y} \in I\!R$ we have

$$\begin{aligned} |\varphi_n(y) - \varphi_n(\hat{y})| &= 2^{-n\alpha}|\zeta(2^n y) - \zeta(2^n \hat{y})| \\ &\leq 2^{-n\alpha}||\zeta||_{0,\alpha}\, |2^n y - 2^n \hat{y}|^\alpha \\ &= ||\zeta||_{0,\alpha}\, |y - \hat{y}|^\alpha, \end{aligned}$$

which shows the uniform boundedness of (φ_n) in $\mathcal{C}^{0,\alpha}(\Gamma_2)$.

Claim 2: $R_1\varphi_n \to 0$ in $\mathcal{C}^0(\Gamma_1 \cap \{|X| \leq \varepsilon_0\})$.
For $|X| \leq \varepsilon_0$, we compute

$$\begin{aligned} R_1\varphi_n(X) &= -2^{-n\alpha} \int_{I\!R} \frac{s\psi_1'(x)}{s^2 + \psi_1(x)^2}\zeta(2^n(s+x))\, ds \\ &= -2^{-n\alpha} \int_{-M}^{M} \frac{s\psi_1'(x)}{s^2 + \psi_1(x)^2} \left[\zeta(2^n(s+x)) - \zeta(2^n x)\right]\, ds, \end{aligned}$$

where M is an upper bound on the support of $\zeta(2^n(\cdot + x))$ which is uniform in n and in $X \in \Gamma_1, |X| \leq \varepsilon_0$. Using the fact that $\zeta \in \mathcal{C}^{0,\alpha/2}(I\!R)$ to control $\zeta(2^n(s+x)) - \zeta(2^n x)$ in the integral above, we obtain

$$\begin{aligned} |R_1\varphi_n(X)| &\leq 2^{-n\alpha/2}\, C\, |\psi_1'(x)| \int_0^M \frac{s^{1+\alpha/2}}{s^2 + \psi_1^2(x)} \\ &\leq 2^{-n\alpha/2}\, C||\psi_1'||_\infty, \end{aligned}$$

which proves the claim.

Claim 3: $\lim_{n\to\infty} \frac{|R_1\varphi_n(X_n) - R_1\varphi_n(0)|}{|X_n - 0|^\alpha} = 0$, where $X_n := (2^{-n}, \psi_1(2^{-n}))$.
Indeed, denoting again by M a bound on the support of $s \to \zeta(2^n s + 1)$ which is uniform in n, we form

$$\begin{aligned} \frac{|R_1\varphi_n(X_n) - R_1\varphi_n(0)|}{2^{-n\alpha}} &= \left| \int_{-M}^{M} \frac{s\psi_1'(2^{-n})}{s^2 + \psi_1(2^{-n})^2}\zeta(2^n s + 1)\, ds \right| \\ &\leq \int_{-M}^{M} \frac{|s\psi_1'(2^{-n})|}{s^2 + \psi_1(2^{-n})^2}\, |\zeta(2^n s + 1) - \zeta(1)|\, ds. \end{aligned}$$

For $s \neq 0$, the integrand is bounded by

$$2||\zeta||_\infty\, |\psi_1'(2^{-n})|\, |s|^{-1} \leq 2||\zeta||_\infty\, ||\psi_1||_{1,\alpha}\, 2^{-n\alpha}|s|^{-1},$$

and so it tends to 0 a.e. Moreover, since $\psi_1(0) = \psi_1'(0) = 0$, the integrand is bounded by

$$|\psi_1'(2^{-n})|\, \frac{|s|}{s^2 + \psi_1(2^{-n})^2}\, ||\zeta||_{0,\alpha}\, 2^{n\alpha}|s|^\alpha \leq ||\psi_1||_{1,\alpha}\, ||\zeta||_{0,\alpha}\, |s|^{\alpha-1},$$

which is integrable on $(0, M)$. The claim then follows from the Lebesgue dominated convergence Theorem.

Claim 4: $R_2\varphi_n \to 0$ in $\mathcal{C}^0(\Gamma_1 \cap \{|X| \le \varepsilon_0\})$.
Indeed, we have for $X = (x, \psi_1(x)) \in \Gamma_1 \cap \{|X| \le \varepsilon_0\}$,

$$\begin{aligned} |R_2\varphi_n(X)| &\le \int_{I\!\!R} \frac{1}{t^2+1} |\varphi_n(t|\psi_1(x)| + x, 0)|\, dt \\ &= 2^{-n\alpha} \int_{I\!\!R} \frac{1}{t^2+1} |\zeta(2^n t|\psi_1(x)| + 2^n x)|\, dt \\ &\le C\, 2^{-n\alpha} \int_{I\!\!R} \frac{1}{1+t^2}, \end{aligned}$$

which proves the claim.

Claim 5: $\lim_{n\to\infty} \frac{|R_2\varphi_n(X_n) - R_2\varphi_n(0)|}{|X_n - 0|^\alpha} = \pi|\zeta(1)| \ne 0$, where $X_n = (2^{-n}, \psi_1(2^{-n}))$.
Indeed, we have

$$\frac{R_2\varphi_n(X_n) - R_2\varphi_n(0)}{|2^{-n} - 0|^\alpha} = \int_{I\!\!R} \frac{\zeta(2^n t|\psi_1(2^{-n})| + 1)}{t^2+1}\, dt.$$

Since ψ_1 has regularity $\mathcal{C}^{1,\alpha}$ and since $\psi_1(0) = \psi_1'(0) = 0$, we see that

$$|2^n \psi_1(2^{-n})| \le C 2^n (2^{-n})^{1+\alpha} = C 2^{-n\alpha},$$

so that as $n \to \infty$,

$$\left\{ \begin{array}{lcl} \frac{1}{t^2+1}\zeta(2^n t|\psi_1(2^{-n})| + 1) & \to & \frac{\zeta(1)}{t^2+1} \quad \text{a.e. } t \in I\!\!R, \\ \left| \frac{1}{t^2+1}\zeta(2^n t|\psi_1(2^{-n})| + 1) \right| & \le & \frac{||\zeta||_\infty}{t^2+1}, \end{array} \right. \tag{91}$$

and the Lebesgue dominated convergence Theorem now shows that

$$\begin{aligned} \frac{|R_2\varphi_n(X_n) - R_2\varphi_n(0)|}{|X_n - 0|^\alpha} &\sim \frac{|R_2\varphi_n(X_n) - R_2\varphi_n(0)|}{|2^{-n} - 0|^\alpha} \\ &\to |\zeta(1)| \int_{I\!\!R} \frac{1}{1+t^2} = \pi\, |\zeta(1)| \ne 0, \end{aligned}$$

as $n \to \infty$, which proves the claim.

We thus have exhibited a sequence $(\varphi_n)_{n\ge 1}$, bounded in $\mathcal{C}^{0,\alpha}(\Gamma_2)$, such that $(R_1+R_2)\varphi_n$ converges to 0 in $\mathcal{C}^0(\Gamma_1 \cap \{|X| \le \varepsilon_0\})$, albeit no subsequence of $(R_1+R_2)\varphi_n$ converges to 0 in $\mathcal{C}^{0,\alpha}(\Gamma_1 \cap \{|X| \le \varepsilon_0\})$. Therefore no subsequence of $(R_1+R_2)\varphi_n$ converges in $\mathcal{C}^{0,\alpha}(\Gamma_1 \cap \{|X| \le \varepsilon_0\})$, and so R_1+R_2 is not a compact operator from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1 \cap \{|X| \le \varepsilon_0\})$. Since $L_2^0 - \frac{1}{2\pi}(R_1+R_2)$ is $\mathcal{C}^{0,\alpha}$ compact, it immediately follows that L_2^0 is not a compact operator from $\mathcal{C}^{0,\alpha}(\Gamma_2)$ to $\mathcal{C}^{0,\alpha}(\Gamma_1)$ for any $\alpha < \alpha_0$.

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